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# A new class of identities involving Cauchy numbers, harmonic numbers and zeta values

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## Abstract

Improving an old idea of Hermite, we associate to each natural number  $k$  a modified zeta function of order  $k$ . The evaluation of the values of these functions  $F_k$  at positive integers reveals a wide class of identities linking Cauchy numbers, harmonic numbers and zeta values.

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## 1 Introduction

It has been well known since the second-half of the 19th century that the Riemann zeta function may be represented by the (normalized) Mellin transform (cf. [14])

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} dt \quad \text{for } \Re(s) > 1,$$

and from late works of Hermite (cf. [11]) that one has also

$$\zeta(s) - \frac{1}{s-1} = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} \left( \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} (1-e^{-t})^n \right) dt \quad \text{for } \Re(s) \geq 1,$$

where  $\lambda_1 = \frac{1}{2}$  and  $\lambda_{n+1} = \int_0^1 x(1-x) \cdots (n-x) dx$  are the (non-alternating) Cauchy numbers<sup>1</sup>.

Improving Hermite's idea, one may, more generally, consider Mellin transforms of type

$$F(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} f(1-e^{-t}) dt \quad \text{with } f(z) = \sum_{n=1}^{\infty} \omega_n \frac{z^n}{n^k}$$

for suitable sequences  $(\omega_n)_{n \geq 1}$  of rational numbers. The simplest interesting case  $\omega_n = 1$  corresponds to the *Arakawa-Kaneko zeta function* and has been studied extensively in [8]. In this article, we investigate the case  $\omega_n = \frac{\lambda_n}{n!}$ , *i.e.*, we study the function

$$F_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} f_k(1-e^{-t}) dt \quad \text{with } f_k(z) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{z^n}{n^k} \quad (k = 0, 1, 2, \dots),$$

which is *a priori* defined in the half-plane  $\Re(s) \geq 1$  but analytically continues in the whole complex  $s$ -plane (Theorem 7). We call this function  $F_k$  the *modified zeta function of order  $k$* . An evaluation by two different methods of the values of  $F_k$  at positive integers  $q$  leads to a new class of identities linking Cauchy numbers, harmonic numbers and zeta values. In the case  $k = 0$ , *Hermite's formula* for  $\zeta$  (cf. [7]) is regained, *i.e.*,

$$F_0(q) = \zeta(q) - \frac{1}{q-1} = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} P_{q-1}(H_n^{(1)}, H_n^{(2)}, \dots, H_n^{(q-1)}),$$

where the polynomials  $P_m$  are the *modified Bell polynomials* defined by the generating function

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{z^k}{k}\right) = \sum_{m=0}^{\infty} P_m(x_1, \dots, x_m) z^m,$$

evaluated at harmonic numbers  $H_n^{(m)} = \sum_{j=1}^n \frac{1}{j^m}$ . In the simplest higher case  $k = 1$ , this extension of Hermite's formula leads to the following new relation (Theorem 10):

$$F_1(q) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^2} P_{q-1}(H_n, H_n^{(2)}, \dots, H_n^{(q-1)}) = \sum_{n=1}^{\infty} \frac{\log(n+1)}{n^q} + \gamma\zeta(q) + \zeta(q+1) - \sum_{n=1}^{\infty} \frac{H_n}{n^q} - \sum_{k=1}^{q-1} \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{(n+1)^k n^{q-k}},$$

where  $H_n = H_n^{(1)}$ , and  $\gamma = \lim_{n \rightarrow \infty} (H_n - \log n)$  is the Euler-Mascheroni constant.

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<sup>1</sup>The sequence of numbers  $\frac{\lambda_n}{n!}$  appeared for the first time in a letter of James Gregory dated back to 1670 (cf. *The correspondence of Isaac Newton*, vol. 1, p. 46). For this reason, they are sometimes called *Gregory coefficients*.

For example, for  $q = 2$ , since  $P_1(H_n) = H_n$  and  $\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3)$  (cf. [6], [7]), then the previous relation may be written

$$F_1(2) = \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n! n^2} = \sum_{n=1}^{\infty} \frac{\log(n+1)}{n^2} + \gamma\zeta(2) - \zeta(3) - 1,$$

and this generalizes the known formula

$$F_0(2) = \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n! n} = \zeta(2) - 1.$$

The function  $F_k$  also has an interesting interpretation in terms of Ramanujan summation (cf. [3]) as underscored by Theorem 11. In particular, one shows the identity

$$F_k(1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{1}{n^{k+1}} = \sum_{n \geq 1}^{\mathcal{R}} \frac{P_k(H_n, H_n^{(2)}, \dots, H_n^{(k)})}{n},$$

where, in the right member,  $\sum_{n \geq 1}^{\mathcal{R}}$  denotes the sum (in the sense of Ramanujan) of the divergent series. This raises a kind of reciprocity between  $F_k(1)$  and  $F_0(k+1)$ .

## 2 Preliminaries

### 2.1 The non-alternating Cauchy numbers

**Definition 1.** The *non-alternating Cauchy numbers* (cf. [7], [12]) are the sequence of (positive) rational numbers  $(\lambda_n)_{n \geq 1}$  defined by the exponential generating function

$$\frac{z}{\log(1-z)} + 1 = \sum_{n \geq 1} \frac{\lambda_n}{n!} z^n. \quad (1)$$

Dividing by  $z$  and setting  $z = 1 - e^{-t}$  and  $t > 0$ , this relation may be rewritten

$$\frac{1}{1 - e^{-t}} - \frac{1}{t} = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} (1 - e^{-t})^{n-1}. \quad (2)$$

From (1), one may easily deduce the following recursive relation

$$\sum_{j=1}^n \frac{\lambda_j}{j!(n-j+1)} - \frac{1}{n+1} = 0 \quad \text{for } n \geq 1.$$

**Example 1.** The first non-alternating Cauchy numbers are

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{6}, \lambda_3 = \frac{1}{4}, \lambda_4 = \frac{19}{30}, \lambda_5 = \frac{9}{4}.$$

## 2.2 The modified Bell polynomials evaluated at harmonic numbers

**Definition 2.** The *modified Bell polynomials* (cf. [5], [7], [10]) are the polynomials  $P_m$  defined for all natural numbers  $m$  by  $P_0 = 1$  and the generating function

$$\exp\left(\sum_{k \geq 1} x_k \frac{z^k}{k}\right) = 1 + \sum_{m \geq 1} P_m(x_1, \dots, x_m) z^m. \quad (3)$$

The general explicit expression for  $P_m$  is

$$P_m(x_1, \dots, x_m) = \sum_{k_1+2k_2+3k_3+\dots=m} \frac{1}{k_1!k_2!k_3!\dots} \left(\frac{x_1}{1}\right)^{k_1} \left(\frac{x_2}{2}\right)^{k_2} \left(\frac{x_3}{3}\right)^{k_3} \dots$$

One may also compute recursively the polynomials  $P_m$  by means of the following relation

$$mP_m(x_1, \dots, x_m) = \sum_{k=1}^m x_k P_{m-k}(x_1, \dots, x_{m-k}) \quad (m \geq 1).$$

**Proposition 1.** For all natural numbers  $m$ , and each integer  $n \geq 1$ ,

$$\int_0^{+\infty} e^{-t} (1 - e^{-t})^{n-1} \frac{t^m}{m!} dt = \frac{P_m(H_n, \dots, H_n^{(m)})}{n}, \quad (4)$$

with

$$H_n^{(m)} = \sum_{j=1}^n \frac{1}{j^m} \quad \text{and} \quad H_n = H_n^{(1)}.$$

*Proof.* One starts from the classical Euler relation (cf. [14])

$$B(a, b) = \int_0^1 u^{a-1} (1-u)^{b-1} du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

and substitute  $u = e^{-t}$ ,  $a = 1 - z$ , and  $b = n + 1$ ; then one obtains

$$\int_0^{+\infty} e^{-t} (1 - e^{-t})^n e^{tz} dt = \frac{n!}{(1-z)(2-z)\dots(n+1-z)}.$$

Moreover, one has

$$\begin{aligned} \frac{n!}{(1-z)(2-z)\dots(n+1-z)} &= \frac{n!}{(n+1)!} \times \prod_{j=0}^n \left(1 - \frac{z}{j+1}\right)^{-1} \\ &= \frac{1}{(n+1)} \times \exp\left(-\sum_{j=0}^n \log\left(1 - \frac{z}{j+1}\right)\right) \\ &= \frac{1}{(n+1)} \times \exp\left(\sum_{j=0}^n \sum_{k=1}^{\infty} \frac{z^k}{k(j+1)^k}\right) \\ &= \frac{1}{(n+1)} \exp\left(\sum_{k=1}^{\infty} H_{n+1}^{(k)} \frac{z^k}{k}\right) \\ &= \sum_{m=0}^{\infty} \frac{P_m(H_{n+1}^{(1)}, \dots, H_{n+1}^{(m)})}{n+1} z^m \quad (\text{by (3)}). \end{aligned}$$

Thus (4) results by identification of the term in  $z^m$ . □

**Example 2.** For small values of  $m$ , one has

$$\begin{aligned} P_1(H_n) &= H_n, \quad P_2(H_n, H_n^{(2)}) = \frac{(H_n)^2}{2} + \frac{H_n^{(2)}}{2}, \\ P_3(H_n, H_n^{(2)}, H_n^{(3)}) &= \frac{(H_n)^3}{6} + \frac{H_n H_n^{(2)}}{2} + \frac{H_n^{(3)}}{3}. \end{aligned}$$

### 2.3 The Laplace-Borel transformation

We consider the vector space  $E$  of complex-valued functions  $f \in \mathcal{C}^1(]0, +\infty[)$  such that

for all  $\varepsilon > 0$ , there exists  $C_\varepsilon > 0$  such that  $|f(t)| \leq C_\varepsilon e^{\varepsilon t}$  for all  $t \in ]0, +\infty[$ .

In particular, a function  $f \in E$  satisfies the following two properties:

- a) for all  $x$  with  $\Re(x) > 0$ ,  $t \mapsto e^{-xt} f(t)$  is integrable on  $]0, +\infty[$ ,
- b) for all  $\beta$  with  $0 < \beta < 1$ ,  $t \mapsto |f(t)| \frac{1}{t^\beta}$  is integrable on  $]0, 1[$ .

We recall now some basic properties (cf. [13]) of the Laplace transformation in this frame which are appropriate for our purpose.

**Definition 3.** Let  $f$  be a function in  $E$ . The *Laplace transform*  $\mathcal{L}(f)$  of  $f$  is defined by

$$\mathcal{L}(f)(x) = \int_0^{+\infty} e^{-xt} f(t) dt \quad \text{for } \Re(x) > 0.$$

**Proposition 2** (cf. [13]). Let  $\mathcal{E} = \mathcal{L}(E)$  be the image of  $E$  under  $\mathcal{L}$ . If  $a$  is a function in  $\mathcal{E}$ , then

- a)  $a$  is an analytic function of  $x$  in the half-plane  $\Re(x) > 0$ ,
- b)  $a(x) \rightarrow 0$  when  $\Re(x) \rightarrow +\infty$ ,
- c)  $\mathcal{L} : E \rightarrow \mathcal{E}$  is an isomorphism.

**Definition 4.** Let  $a \in \mathcal{E}$ . The *Borel transform* of  $a$  is the unique function  $\hat{a} \in E$  such that  $a = \mathcal{L}(\hat{a})$ . One has the two reciprocal formulas

$$\hat{a}(t) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} e^{zt} a(z) dz \quad \text{for all } c > 0 \text{ and } t > 0,$$

and

$$a(x) = \int_0^{+\infty} e^{-xt} \hat{a}(t) dt \quad \text{for } \Re(x) > 0.$$

**Definition 5.** Let  $f$  and  $g$  be two functions in  $E$ . The *convolution product*  $f * g$  of  $f$  and  $g$  is the function defined for all  $t > 0$  by

$$(f * g)(t) = \int_0^t f(u)g(t-u) du.$$

**Proposition 3** (cf. [13]). If  $f \in E$  and  $g \in E$ , then  $f * g \in E$  and

$$\mathcal{L}(f * g) = \mathcal{L}(f) \mathcal{L}(g). \quad (5)$$

Hence, if  $a \in \mathcal{E}$  and  $b \in \mathcal{E}$  then  $ab \in \mathcal{E}$  since  $ab = \mathcal{L}(\widehat{a} * \widehat{b})$ .

**Theorem 1.** Let  $a$  be a function in  $\mathcal{E}$ . Then the series

$$\sum_{n \geq 1} \frac{\lambda_n}{n!} \int_0^{+\infty} e^{-t}(1 - e^{-t})^{n-1} \widehat{a}(t) dt$$

converges and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \int_0^{+\infty} e^{-t}(1 - e^{-t})^{n-1} \widehat{a}(t) dt = \int_0^{+\infty} \left( \frac{1}{1 - e^{-t}} - \frac{1}{t} \right) e^{-t} \widehat{a}(t) dt. \quad (6)$$

*Proof.* By (2)

$$\int_0^{+\infty} \left( \frac{1}{1 - e^{-t}} - \frac{1}{t} \right) e^{-t} \widehat{a}(t) dt = \int_0^{+\infty} \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} (1 - e^{-t})^{n-1} e^{-t} \widehat{a}(t) dt.$$

In the right member, the order of  $\int_0^{+\infty}$  and  $\sum_{n=1}^{\infty}$  may be interchanged since

$$\begin{aligned} \int_0^{+\infty} \sum_{n=1}^{\infty} \left| \frac{\lambda_n}{n!} (1 - e^{-t})^{n-1} e^{-t} \widehat{a}(t) \right| dt &= \int_0^{+\infty} \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} (1 - e^{-t})^{n-1} e^{-t} |\widehat{a}(t)| dt \\ &= \int_0^{+\infty} \left( \frac{1}{1 - e^{-t}} - \frac{1}{t} \right) e^{-t} |\widehat{a}(t)| dt, \end{aligned}$$

and the convergence of this last integral follows from the assumption that  $a \in \mathcal{E}$ . □

**Example 3.** Let  $a(x) = \frac{1}{x^s}$  with  $\Re(s) \geq 1$ . Then  $a \in \mathcal{E}$  and  $\widehat{a}(t) = \frac{t^{s-1}}{\Gamma(s)}$ . Hence

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \int_0^{+\infty} e^{-t}(1 - e^{-t})^{n-1} \frac{t^{s-1}}{\Gamma(s)} dt &= \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-t} \left( \frac{1}{1 - e^{-t}} - \frac{1}{t} \right) t^{s-1} dt \\ &= \begin{cases} \gamma & \text{if } s = 1, \\ \zeta(s) - \frac{1}{s-1} & \text{if } s \neq 1 \end{cases} \end{aligned}$$

where  $\gamma$  refers to the Euler constant. In particular, since

$$\int_0^{+\infty} e^{-t}(1 - e^{-t})^{n-1} dt = \frac{1}{n} \quad \text{for each integer } n \geq 1,$$

then

$$\gamma = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{1}{n}.$$

### 3 The operator $D$

**Proposition 4.** If  $a \in \mathcal{E}$ , then the integral

$$\int_0^{+\infty} e^{-t}(1 - e^{-t})^{x-1} \widehat{a}(t) dt$$

converges for all  $x$  with  $\Re(x) > 0$ .

*Proof.* If  $a \in \mathcal{E}$  and  $\Re(x) > 0$ , we may write for  $t \in ]0, +\infty[$ ,

$$\left| e^{-t}(1 - e^{-t})^{x-1} \widehat{a}(t) \right| \leq e^{-t} e^{(1-\Re(x))(-\log(1-e^{-t}))} |\widehat{a}(t)| .$$

The convergence when  $t \rightarrow +\infty$  results from the inequality

$$e^{-t} e^{(1-\Re(x))(-\log(1-e^{-t}))} |\widehat{a}(t)| \leq \frac{e^{-t}}{1 - e^{-t}} |\widehat{a}(t)| \leq 2e^{-t} |\widehat{a}(t)| \quad (\text{for } t \geq \log 2).$$

The convergence when  $t \rightarrow 0$  results from the inequality

$$e^{(1-\Re(x))(-\log(1-e^{-t}))} \leq \begin{cases} 1 & \text{if } \Re(x) \geq 1, \\ \frac{1}{(1-e^{-t})^{(1-\Re(x))}} & \text{if } 0 < \Re(x) < 1 \end{cases}$$

since the function  $t \mapsto e^{-t} |\widehat{a}(t)| \frac{1}{(1 - e^{-t})^\beta}$  is integrable at 0 for  $0 < \beta < 1$  by the definition of  $E$  (note that  $(1 - e^{-t})^{-\beta} \leq (kt)^{-1}$  for small enough  $t$ ).  $\square$

**Definition 6.** Let  $a$  be a function in  $\mathcal{E}$ . We call  $D(a)$  the function defined for all  $x$  with  $\Re(x) > 0$  by

$$D(a)(x) = \int_0^{+\infty} e^{-t}(1 - e^{-t})^{x-1} \widehat{a}(t) dt. \quad (7)$$

**Remark 1.** a) By Theorem 1, the series  $\sum_{n \geq 1} \frac{\lambda_n}{n!} D(a)(n)$  converges and its sum is given by formula (6).

b) The values of  $D(a)$  at positive integers may be computed directly without recourse to  $\widehat{a}$ . The development of  $(1 - e^{-t})^n$  by the binomial theorem gives

$$D(a)(n+1) = \sum_{k=0}^n (-1)^k \binom{n}{k} a(k+1) \quad \text{for all integer } n \geq 0. \quad (8)$$

**Definition 7.** We call  $\Lambda$  the  $C^1$ -diffeomorphism of  $\mathbb{R}_+$  defined by  $\Lambda(u) = -\log(1 - e^{-u})$ . In particular, it is important to note that  $\Lambda$  is involutive:

$$\Lambda^{-1} = \Lambda.$$



**Theorem 2.** *Let  $a$  be a function in  $\mathcal{E}$ . Then the function  $D(a) \in \mathcal{E}$  and, moreover, verifies the relation*

$$\widehat{D(a)} = \widehat{a}(\Lambda), \quad (9)$$

where  $\widehat{a}(\Lambda)$  denotes  $\widehat{a} \circ \Lambda$ .

*Proof.* The change of variables  $t = \Lambda(u)$  in (7) gives

$$D(a)(x) = \int_0^{+\infty} e^{-xu} \widehat{a}(\Lambda(u)) du \quad \text{for } \Re(x) > 0.$$

Thus,  $D(a) = \mathcal{L}(\widehat{a}(\Lambda))$ . It remains to prove that  $D(a) \in \mathcal{E}$ . One has only to check that the function  $\widehat{a}(\Lambda)$  is in  $E$ . This function being in  $\mathcal{C}^1(]0, +\infty[)$ , it suffices to show that for all  $\varepsilon > 0$ , the function  $u \mapsto e^{-\varepsilon u} |\widehat{a}(-\log(1 - e^{-u}))|$  is bounded on  $]0, +\infty[$ . This results from the existence of  $C_\varepsilon > 0$  such that

$$|\widehat{a}(-\log(1 - e^{-u}))| \leq C_\varepsilon (1 - e^{-u})^\varepsilon \quad \text{for all } u \in ]0, +\infty[.$$

□

**Example 4.** Let  $a(x) = \frac{1}{x^s}$  with  $\Re(s) \geq 1$ . Then  $\widehat{a}(t) = \frac{t^{s-1}}{\Gamma(s)}$ . Thus, by (9),

$$D\left(\frac{1}{x^s}\right) = \mathcal{L}\left(\frac{\Lambda^{s-1}}{\Gamma(s)}\right), \quad (10)$$

and if  $s = m + 1$  with  $m$  a natural number and  $n \geq 1$ , then by (4),

$$D\left(\frac{1}{x^{m+1}}\right)(n) = \frac{P_m(H_n, \dots, H_n^{(m)})}{n}. \quad (11)$$

**Remark 2.** Theorem 2 may be summarized in the following diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{D} & \mathcal{E} \\ \downarrow \mathcal{L}^{-1} & & \uparrow \mathcal{L} \\ E & \xrightarrow{\Lambda^*} & E \end{array}$$

where  $\Lambda^*(\widehat{a}) = \widehat{a}(\Lambda)$ . The algebraic properties of  $D$  are summed up in the following theorem.

**Theorem 3.** *The operator  $D$  is an automorphism of  $\mathcal{E}$  which verifies  $D = D^{-1}$  and lets the function  $x \mapsto \frac{1}{x}$  invariant.*

*Proof.* We can write  $D = \mathcal{L}\Lambda^*\mathcal{L}^{-1}$  and  $\Lambda^*$  is an automorphism of  $E$  which verifies  $\Lambda^* = (\Lambda^*)^{-1}$  since  $\Lambda = \Lambda^{-1}$ . Furthermore,

$$D\left(\frac{1}{x}\right) = \mathcal{L}(1) = \frac{1}{x}.$$

□

## 4 The harmonic product

Our aim is to define the harmonic product of two functions  $a$  and  $b$  in  $\mathcal{E}$  as being the unique function  $f$  of  $\mathcal{E}$  such that

$$D(a)(x).D(b)(x) = D(f)(x).$$

Thus, we have to establish that such a function exists and is unique. In order to do this, we introduce first a  $\Lambda$ -convolution product of two functions in  $E$ .

### 4.1 The $\Lambda$ -convolution product

**Proposition 5.** If  $a$  and  $b$  are in  $\mathcal{E}$ , then  $\widehat{a}(\Lambda) * \widehat{b}(\Lambda) \in E$ .

*Proof.* From the definition of the convolution product, one may write

$$\left(\widehat{a}(\Lambda) * \widehat{b}(\Lambda)\right)(t) = \int_0^t \widehat{a}(\Lambda(u))\widehat{b}(\Lambda(t-u))du.$$

Now, for all  $\varepsilon > 0$ , there exists  $C_\varepsilon > 0$  and  $D_\varepsilon > 0$  such that

$$\begin{aligned} |\widehat{a}(-\log(1 - e^{-u}))| &\leq C_\varepsilon(1 - e^{-u})^\varepsilon \text{ and} \\ |\widehat{b}(-\log(1 - e^{-(t-u)}))| &\leq D_\varepsilon(1 - e^{-(t-u)})^\varepsilon \text{ for all } u \in ]0, +\infty[. \end{aligned}$$

It follows that

$$\left|(\widehat{a}(\Lambda) * \widehat{b}(\Lambda))(t)\right| \leq C_\varepsilon D_\varepsilon \int_0^t (1 - e^{-u})^\varepsilon (1 - e^{-(t-u)})^\varepsilon du.$$

One has also

$$\begin{aligned} \int_0^t (1 - e^{-u})^\varepsilon (1 - e^{-(t-u)})^\varepsilon du &= (1 - e^{-t})^{1+2\varepsilon} \int_0^1 u^\varepsilon (1 - u)^\varepsilon \frac{1}{(1 - (1 - e^{-t})u)^{\varepsilon+1}} du \\ &\leq (1 - e^{-t})^{1+2\varepsilon} \int_0^1 \frac{1}{(1 - (1 - e^{-t})u)^{\varepsilon+1}} du \leq (1 - e^{-t})^{1+2\varepsilon} \frac{e^{t\varepsilon} - 1}{(1 - e^{-t})^\varepsilon} \\ &\leq (1 - e^{-t})^{2\varepsilon} \frac{e^{t\varepsilon} - 1}{\varepsilon} \leq \frac{e^{t\varepsilon}}{\varepsilon}. \end{aligned}$$

Hence,  $\left|(\widehat{a}(\Lambda) * \widehat{b}(\Lambda))(t)\right| \leq C_\varepsilon D_\varepsilon \frac{e^{t\varepsilon}}{\varepsilon}$ , which proves that this function belongs to  $E$  as required.  $\square$

**Definition 8.** Let  $a$  and  $b$  be two functions in  $\mathcal{E}$ . The  $\Lambda$ -convolution product  $\widehat{a} \circledast \widehat{b}$  of  $\widehat{a}$  and  $\widehat{b}$  is defined by

$$\widehat{a} \circledast \widehat{b} = \Lambda^*(\Lambda^*(\widehat{a}) * \Lambda^*(\widehat{b})),$$

or equivalently (since  $\Lambda^* = (\Lambda^*)^{-1}$ )

$$(\widehat{a} \circledast \widehat{b})(\Lambda) = \widehat{a}(\Lambda) * \widehat{b}(\Lambda).$$

**Remark 3.** The  $\Lambda$ -convolution product inherits the algebraic properties of the ordinary convolution product, *i.e.*, bilinearity, commutativity, and associativity.

## 4.2 The harmonic product

**Definition 9.** Let  $a$  and  $b$  two functions in  $\mathcal{E}$ . The *harmonic product*  $a \bowtie b$  of  $a$  and  $b$  is defined by

$$a \bowtie b = \mathcal{L}(\widehat{a} \circledast \widehat{b}) \in \mathcal{E}.$$

This construction may be summarized in the following diagram:

$$\begin{array}{ccccc} (a, b) & \longrightarrow & (\widehat{a}, \widehat{b}) & \longrightarrow & (\widehat{a}(\Lambda), \widehat{b}(\Lambda)) \\ \downarrow & & \downarrow & & \downarrow \\ a \bowtie b & \longleftarrow & \widehat{a} \circledast \widehat{b} & \longleftarrow & \widehat{a}(\Lambda) * \widehat{b}(\Lambda) \end{array}$$

**Remark 4.** The harmonic product inherits the properties of the  $\Lambda$ -convolution product: it is bilinear, commutative and associative.

**Theorem 4.** Let  $a$  and  $b$  be in  $\mathcal{E}$ . Then,

$$D(a \bowtie b) = D(a) D(b), \quad (12)$$

and

$$D(ab) = D(a) \bowtie D(b). \quad (13)$$

*Proof.* One knows from Theorem 2 that

$$D = \mathcal{L}\Lambda^* \mathcal{L}^{-1}.$$

Hence

$$D(a \bowtie b) = \mathcal{L}\Lambda^* \mathcal{L}^{-1}(a \bowtie b) = \mathcal{L}\Lambda^*(\widehat{a} \circledast \widehat{b}) = \mathcal{L}(\Lambda^*(\widehat{a}) * \Lambda^*(\widehat{b})),$$

and it follows from (5) and (9) that

$$\mathcal{L}(\Lambda^*(\widehat{a}) * \Lambda^*(\widehat{b})) = \mathcal{L}(\Lambda^*(\widehat{a})) \mathcal{L}(\Lambda^*(\widehat{b})) = D(a) D(b)$$

which proves (12). Moreover, (12) enables us to write

$$D(D(a) \bowtie D(b)) = D^2(a) D^2(b) = ab \quad (\text{since } D = D^{-1}),$$

and so

$$D(ab) = D^2(D(a) \bowtie D(b)) = D(a) \bowtie D(b)$$

which proves (13). □

**Remark 5.** The values of  $(a \bowtie b)(n)$  may be computed without recourse to  $\widehat{a}$  and  $\widehat{b}$ . By elementary transformations, it can be shown that

$$(a \bowtie b)(n+1) = \int_0^{+\infty} \int_0^{+\infty} (e^{-t-s})(e^{-t} + e^{-s} - e^{-t}e^{-s})^n \widehat{a}(t) \widehat{b}(s) dt ds.$$

Hence, if the numbers  $C_n^{k,l}$  are defined by

$$(X + Y - XY)^n = \sum_{\substack{0 \leq k \leq n \\ 0 \leq l \leq n}} C_n^{k,l} X^k Y^l,$$

then one has the following explicit formula:

$$(a \bowtie b)(n+1) = \sum_{\substack{0 \leq k \leq n \\ 0 \leq l \leq n}} C_n^{k,l} a(k+1)b(l+1),$$

which can be rewritten in the following equivalent form:

$$(a \bowtie b)(n+1) = \sum_{0 \leq l \leq k \leq n} (-1)^{k-l} \binom{n}{k} \binom{k}{l} a(k+1)b(n+1-l) \quad (n \geq 0).$$

For small values of  $n$ , this enables one to compute

$$\begin{aligned} (a \bowtie b)(1) &= a(1)b(1), \\ (a \bowtie b)(2) &= a(2)b(1) + a(1)b(2) - a(2)b(2), \\ (a \bowtie b)(3) &= a(3)b(1) + a(1)b(3) + 2a(2)b(2) - 2a(3)b(2) - 2a(2)b(3) + a(3)b(3). \end{aligned}$$

**Theorem 5.** *Let*

$$\left(\frac{1}{x}\right)^{\bowtie k} = \underbrace{\frac{1}{x} \bowtie \frac{1}{x} \bowtie \cdots \bowtie \frac{1}{x}}_k \quad (k = 1, 2, 3, \dots),$$

where  $\frac{1}{x}$  denotes (improperly) the function  $x \mapsto \frac{1}{x}$ . Then, for all natural numbers  $m \geq 0$ ,

$$\left(\frac{1}{x}\right)^{\bowtie(m+1)} = D\left(\frac{1}{x^{m+1}}\right).$$

In particular, for all integers  $n \geq 1$ ,

$$\left(\frac{1}{x}\right)^{\bowtie(m+1)}(n) = \frac{P_m(H_n, \dots, H_n^{(m)})}{n}. \quad (14)$$

*Proof.* By (13) we have

$$D\left(\frac{1}{x^{m+1}}\right) = D\left(\underbrace{\frac{1}{x} \cdots \frac{1}{x}}_{m+1}\right) = \left(D\left(\frac{1}{x}\right)\right)^{\bowtie(m+1)} = \left(\frac{1}{x}\right)^{\bowtie(m+1)} \text{ since } D\left(\frac{1}{x}\right) = \frac{1}{x}.$$

Thus, (14) results from (11). □

### 4.3 The harmonic property

The following theorem explains the main reason why the harmonic product is called ‘harmonic’.

**Theorem 6.** *Let  $a \in \mathcal{E}$ . Then*

$$\frac{1}{x} \bowtie a = \frac{A(x)}{x},$$

where  $A$  denotes the function defined for  $\Re(x) > 0$  by

$$A(x) = \int_0^{+\infty} \frac{e^{-xt} - 1}{e^{-t} - 1} e^{-t} \widehat{a}(t) dt.$$

In particular, for each integer  $n \geq 1$ ,

$$\left( \frac{1}{x} \bowtie a \right) (n) = \frac{A(n)}{n} = \frac{1}{n} \left( \sum_{k=1}^n a(k) \right). \quad (15)$$

*Proof.* By the definition of the harmonic product, one has

$$\frac{1}{x} \bowtie a = \mathcal{L}(1 \circledast \widehat{a}).$$

Now

$$(1 \circledast \widehat{a})(\Lambda(u)) = (1 * \widehat{a}(\Lambda))(u) = \int_0^u \widehat{a}(\Lambda(v)) dv = - \int_{+\infty}^{\Lambda(u)} \widehat{a}(t) \frac{e^{-t}}{1 - e^{-t}} dt$$

(by the change of variables  $t = \Lambda(v)$ ). Hence,

$$(1 \circledast \widehat{a})(u) = \int_u^{+\infty} \widehat{a}(t) \frac{e^{-t}}{1 - e^{-t}} dt.$$

Thus, we have

$$\begin{aligned} \frac{1}{x} \bowtie a &= \int_0^{+\infty} e^{-xu} \left( \int_u^{+\infty} \widehat{a}(t) \frac{e^{-t}}{1 - e^{-t}} dt \right) du \\ &= \int_0^{+\infty} \left( \int_0^t e^{-xu} du \right) \widehat{a}(t) \frac{e^{-t}}{1 - e^{-t}} dt \\ &= \frac{1}{x} \int_0^{+\infty} (1 - e^{-xt}) \widehat{a}(t) \frac{e^{-t}}{1 - e^{-t}} dt \\ &= \frac{A(x)}{x}. \end{aligned}$$

Furthermore, for each integer  $n \geq 1$ , we have

$$A(n) = \int_0^{+\infty} \frac{e^{-nt} - 1}{e^{-t} - 1} e^{-t} \widehat{a}(t) dt = \sum_{k=1}^n a(k).$$

□

**Example 5.**

$$\frac{1}{x} \bowtie \frac{1}{x} = D\left(\frac{1}{x^2}\right) = \mathcal{L}(\Lambda) = \frac{H(x)}{x} \quad \text{with } H(x) = \psi(x+1) + \gamma,$$

$\psi$  denoting the logarithmic derivative of  $\Gamma$ . In particular, for each integer  $n \geq 1$ ,

$$\left(\frac{1}{x} \bowtie \frac{1}{x}\right)(n) = \frac{H(n)}{n} = \frac{H_n}{n}.$$

**Example 6.** For  $\Re(s) \geq 1$ ,

$$\frac{1}{x} \bowtie \frac{1}{x^s} = \frac{H^{(s)}(x)}{x},$$

with

$$H^{(s)}(x) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{1 - e^{-xt}}{1 - e^{-t}} e^{-t} t^{s-1} dt.$$

For each integer  $n \geq 1$ ,

$$\left(\frac{1}{x} \bowtie \frac{1}{x^s}\right)(n) = \frac{H^{(s)}(n)}{n} = \frac{H_n^{(s)}}{n} = \frac{1}{n} \left(\sum_{m=1}^n \frac{1}{m^s}\right).$$

From (15), by induction on  $k$ , we deduce the following important corollary.

**Corollary 1.** For each integer  $k \geq 2$ ,

$$\left(\left(\frac{1}{x}\right)^{\bowtie k} \bowtie a\right)(n) = \frac{1}{n} \left(\sum_{n \geq n_1 \geq \dots \geq n_k \geq 1} \frac{a(n_k)}{n_1 \dots n_{k-1}}\right). \quad (16)$$

**Example 7.** Applying (16) with  $a(x) = \frac{1}{x}$  (and  $k = m$ ), we get

$$\left(\frac{1}{x}\right)^{\bowtie(m+1)}(n) = \frac{1}{n} \left(\sum_{n \geq n_1 \geq \dots \geq n_m \geq 1} \frac{1}{n_1 \dots n_m}\right). \quad (17)$$

Hence, it follows from (14) and (17) that

$$P_m(H_n, H_n^{(2)}, \dots, H_n^{(m)}) = \sum_{n \geq n_1 \geq \dots \geq n_m \geq 1} \frac{1}{n_1 \dots n_m}, \quad (18)$$

which is a nice reformulation of Dilcher's formula (cf. [2], [9]).

## 5 The modified zeta function $F_k$

### 5.1 Integral representation

**Definition 10.** For all  $s \in \mathbb{C}$  with  $\Re(s) \geq 1$  and each natural number  $k$ , the *modified zeta function of order  $k$*  is defined by

$$F_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1 - e^{-t}} f_k(1 - e^{-t}) dt \quad \text{with} \quad f_k(z) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{z^n}{n^k}. \quad (19)$$

**Remark 6.** By (2) and Example 3, one has  $F_0(s) = \zeta(s) - \frac{1}{s-1}$ .

The fact that  $F_k$  may be represented by a Mellin transform enables us to analytically continue this function outside its half-plane of definition by a standard analytic method (cf. [14] section 6.7).

**Theorem 7.** *The function  $F_k$  analytically continues in the whole complex plane as an entire function.*

*Proof.* The function  $z \mapsto \frac{1}{\log(1-z)} + \frac{1}{z}$  being analytic in the disc  $D(0,1)$  with a singularity at 1, we deduce from (1) that the radius of convergence of the series  $\sum_{n=1}^{\infty} \frac{\lambda_n z^n}{n!}$  is equal to 1. Thus 1 is also the radius of convergence of the series  $\sum_{n=1}^{\infty} \frac{\lambda_n z^n}{n! n^k}$  which defines an analytic function  $f_k$  in the disc  $D(0,1)$ . Hence, the function

$$g_k : t \mapsto f_k(1 - e^{-t})$$

is analytic for all  $t \in \mathbb{C}$  such that  $1 - e^{-t} \in D(0,1)$ . Since  $1 - e^0 = 0$ , it follows that  $g_k$  is analytic in a neighbourhood of 0. Since  $g_k(0) = 0$ , the function  $t \mapsto g_k(t) \frac{e^{-t}}{1 - e^{-t}}$  is itself analytic in a neighbourhood of 0. It follows that its Mellin transform analytically continues in the complex plane with simple poles at negative integers which are all cancelled by the poles of  $\Gamma$ .  $\square$

**Theorem 8.** *For all  $s$  with  $\Re(s) > 1$  and each integer  $k \geq 1$ ,*

$$F_k(s) = \vartheta(k)\zeta(s) + \sum_{j=1}^k (-1)^j \vartheta(k-j) Z_j(s) + (-1)^k \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1 - e^{-t}} T^k \left( \frac{e^{-t} - 1}{t} \right) dt \quad (20)$$

with

$$\vartheta(k) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{1}{n^k}, \quad (21)$$

$$Z_j(s) = \sum_{n > n_1 > n_2 > \dots > n_j > 0} \frac{1}{n^s n_1 n_2 \dots n_j}, \quad (22)$$

$$Tf(t) = \int_t^{+\infty} \frac{e^{-u}}{1 - e^{-u}} f(u) du. \quad (23)$$

*Proof.* Formula (20) results from the integral representation (19) and the two following lemmas.

**Lemma 1.** For all  $t > 0$ ,

$$f_k(1 - e^{-t}) = \sum_{j=0}^k (-1)^j \vartheta(k-j) \frac{\Lambda^j(t)}{j!} + (-1)^k T^k \left( \frac{e^{-t} - 1}{t} \right),$$

where  $\vartheta$  is defined by (21) and  $T$  is the operator defined by (23).

*Proof.* Let  $g_k(t) = f_k(1 - e^{-t})$ . The function  $g_k$  verifies the recursive relation

$$g'_k(t) = e^{-t} f'_k(1 - e^{-t}) = \frac{e^{-t}}{1 - e^{-t}} f_{k-1}(1 - e^{-t}) = \frac{e^{-t}}{1 - e^{-t}} g_{k-1}(t).$$

Thus

$$g_k(t) = \int_0^t \frac{e^{-u}}{1 - e^{-u}} g_{k-1}(u) du = g_k(+\infty) - \int_t^{+\infty} \frac{e^{-u}}{1 - e^{-u}} g_{k-1}(u) du$$

with

$$g_k(+\infty) = f_k(1) = \vartheta(k).$$

Thus, one has

$$g_k(t) = \vartheta(k) - \int_t^{+\infty} \frac{e^{-u}}{1 - e^{-u}} g_{k-1}(u) du = \vartheta(k) - T(g_{k-1}),$$

and a repeated iteration  $k$  times of this relation gives

$$g_k(t) = \sum_{j=0}^{k-1} \vartheta(k-j) (-1)^j T^j(1) + (-1)^k T^k(g_0).$$

Now, by (2),

$$g_0(t) = \sum_{n=1}^{\infty} \frac{\lambda_n (1 - e^{-t})^n}{n!} = \frac{e^{-t} - 1}{t} + 1,$$

and thus

$$T^k(g_0) = T^k\left(\frac{e^{-t} - 1}{t}\right) + T^k(1).$$

Hence

$$g_k(t) = \sum_{j=0}^{k-1} \vartheta(k-j) (-1)^j T^j(1) + (-1)^k T^k(1) + (-1)^k T^k\left(\frac{e^{-t} - 1}{t}\right).$$

Since  $\vartheta(0) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} = 1$  (by (1) and a tauberian theorem), one deduces that

$$g_k(t) = \sum_{j=0}^k \vartheta(k-j) (-1)^j T^j(1) + (-1)^k T^k\left(\frac{e^{-t} - 1}{t}\right),$$

and, now, it remains to prove that

$$\frac{\Lambda^j(t)}{j!} = T^j(1),$$

which follows from the recursive relation

$$\frac{\Lambda^j(t)}{j!} = - \int_{+\infty}^t \frac{e^{-u}}{1 - e^{-u}} \frac{\Lambda^{j-1}(u)}{(j-1)!} du = T\left(\frac{\Lambda^{j-1}}{(j-1)!}\right).$$

□



**Lemma 2.** Let  $Z_j(s)$  be defined by (22). Then, for all  $s \in \mathbb{C}$  with  $\Re(s) > 1$ ,

$$Z_j(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} \frac{\Lambda^j(t)}{j!} dt.$$

*Proof.* From the recursive relation

$$\partial \frac{\Lambda^j(t)}{j!} = \frac{\Lambda^{j-1}(t)}{(j-1)!} \partial \Lambda(t) = -\frac{e^{-t}}{1-e^{-t}} \frac{\Lambda^{j-1}(t)}{(j-1)!} = -\sum_{m>0} e^{-mt} \frac{\Lambda^{j-1}(t)}{(j-1)!},$$

and  $\Lambda(t) = \sum_{n>0} \frac{e^{-nt}}{n}$ , one may check by induction on  $j$  that

$$\frac{\Lambda^j(t)}{j!} = \sum_{n_1>n_2>\dots>n_j>0} \frac{e^{-n_1 t}}{n_1} \frac{1}{n_2} \dots \frac{1}{n_j}.$$

Furthermore, one has

$$\frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} e^{-Nt} \frac{e^{-t}}{1-e^{-t}} dt = \sum_{n>N} \frac{1}{n^s} \quad (\text{for } \Re(s) > 1).$$

Hence

$$\frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} \frac{\Lambda^j(t)}{j!} dt = \sum_{n>n_1>n_2>\dots>n_j>0} \frac{1}{n^s} \frac{1}{n_1} \frac{1}{n_2} \dots \frac{1}{n_j} = Z_j(s).$$

□

□

## 5.2 Values of $F_k$ at integers

**Theorem 9.** For all  $s$  in  $\mathbb{C}$  with  $\Re(s) \geq 1$  and each natural number  $k$ , then

$$F_k(s) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^k} D\left(\frac{1}{x^s}\right)(n). \quad (24)$$

In particular, for all natural numbers  $m$ ,

$$F_k(m+1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{P_m(H_n, H_n^{(2)}, \dots, H_n^{(m)})}{n^{k+1}}. \quad (25)$$

*Proof.* The change of variables  $t = \Lambda(u)$  in (19) enables to write

$$F_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} f_k(e^{-u}) (\Lambda(u))^{s-1} du.$$

Since  $D\left(\frac{1}{x^s}\right) = \mathcal{L}\left(\frac{\Lambda^{s-1}}{\Gamma(s)}\right)$ , we deduce (24) from this last expression of  $F_k(s)$ . Moreover,

by (11), one also has  $D\left(\frac{1}{x^{m+1}}\right)(n) = \frac{P_m(H_n, \dots, H_n^{(m)})}{n}$ , which proves (25). □

**Corollary 2.** Let  $\vartheta(s)$  be the Dirichlet series defined for  $\Re(s) > 0$  by

$$\vartheta(s) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{1}{n^s}.$$

Then for each natural number  $k$ ,

$$\vartheta(k+1) = F_k(1). \quad (26)$$

**Example 8.**

$$\begin{aligned} F_0(1) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} = \gamma = \vartheta(1), \\ F_0(2) &= \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!n} = \zeta(2) - 1, \\ F_0(3) &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^2}{n!n} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!n} = \zeta(3) - \frac{1}{2}, \\ F_1(1) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^2} = \vartheta(2), \\ F_1(2) &= \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!n^2}, \\ F_1(3) &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^2}{n!n^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!n^2}. \end{aligned}$$

### 5.3 Identities linking Cauchy numbers, harmonic numbers and zeta values

**Theorem 10.** For all integers  $q \geq 2$ ,

$$\begin{aligned} F_1(q) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^2} P_{q-1}(H_n, H_n^{(2)}, \dots, H_n^{(q-1)}) = \\ &= \sum_{n=1}^{\infty} \frac{\log(n+1)}{n^q} + \gamma \zeta(q) + \zeta(q+1) - \sum_{n=1}^{\infty} \frac{H_n}{n^q} - \sum_{k=1}^{q-1} \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{(n+1)^k n^{q-k}}. \end{aligned} \quad (27)$$

*Proof.* By (20) and (25), one may write

$$\begin{aligned} F_k(q) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^{k+1}} P_{q-1}(H_n, H_n^{(2)}, \dots, H_n^{(q-1)}) = \\ &= \vartheta(k) \zeta(q) + \sum_{j=1}^k (-1)^j \vartheta(k-j) Z_j(q) + (-1)^k \frac{1}{\Gamma(q)} \int_0^{+\infty} t^{q-1} \frac{e^{-t}}{1-e^{-t}} T^k \left( \frac{e^{-t}-1}{t} \right) dt. \end{aligned} \quad (28)$$

We apply now (28) with  $k = 1$ . This gives

$$F_1(q) = \gamma\zeta(q) - \sum_{n \geq 1} \frac{H_{n-1}}{n^q} + \frac{1}{\Gamma(q)} \int_0^{+\infty} t^{q-1} \frac{e^{-t}}{1 - e^{-t}} E_1(t) dt,$$

with  $E_1(t) = -\text{Ei}(-t) = \int_t^{+\infty} \frac{e^{-u}}{u} du$ . Thus

$$F_1(q) = \gamma\zeta(q) - \sum_{n \geq 1} \frac{H_n}{n^q} + \zeta(q+1) + I(q),$$

where

$$I(q) = \frac{1}{\Gamma(q)} \int_0^{+\infty} t^{q-1} \frac{e^{-t}}{1 - e^{-t}} E_1(t) dt = \frac{1}{\Gamma(q)} \sum_{n=1}^{\infty} \int_0^{+\infty} e^{-nt} t^{q-1} E_1(t) dt.$$

Since

$$E_1(t) = -\gamma - \log t + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{t^n}{n!},$$

and  $-\gamma - \log t = \widehat{\frac{\log x}{x}}$  (cf. [13]), then  $E_1 = \widehat{\frac{\log(x+1)}{x}}$ . Thus

$$\int_0^{+\infty} e^{-nt} t^{q-1} E_1(t) dt = (-1)^{q-1} \left( \frac{\log(x+1)}{x} \right)^{(q-1)} (n).$$

Hence, by a calculation of the  $(q-1)$ th derivative, we get

$$I(q) = \frac{(-1)^{q-1}}{(q-1)!} \sum_{n=1}^{\infty} \left( \frac{\log(x+1)}{x} \right)^{(q-1)} (n) = \sum_{n=1}^{\infty} \frac{\log(n+1)}{n^q} - \sum_{k=1}^{q-1} \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{(n+1)^k n^{q-k}}.$$

□

**Remark 7.** 1) We recall *Euler's formula* (cf. [6])

$$\sum_{n=1}^{\infty} \frac{H_n}{n^q} = \begin{cases} \frac{1}{2}(q+2)\zeta(q+1) - \frac{1}{2} \sum_{k=1}^{q-2} \zeta(k+1)\zeta(q-k) & \text{for } q > 2, \\ 2\zeta(3) & \text{for } q = 2. \end{cases}$$

2) From  $\sum_{n=1}^{\infty} \frac{1}{(n+1)n} = 1$ , and the decomposition

$$\frac{1}{(n+1)^k n^{q-k}} = \frac{1}{(n+1)^{k-1} n^{q-k}} - \frac{1}{(n+1)^k n^{q-k-1}} \quad (0 < k < q),$$

the sum of the series  $\sum_{n=1}^{\infty} \frac{1}{(n+1)^k n^{q-k}}$  may be expressed as a linear combination of zeta values and integers.

**Example 9.**

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\log(n+1)}{n^2} + \gamma\zeta(2) - \zeta(3) - 1 &= \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!n^2}, \\
\sum_{n=1}^{\infty} \frac{\log(n+1)}{n^3} + \gamma\zeta(3) - \frac{1}{10}\zeta(2)^2 - \frac{1}{2}\zeta(2) &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^2}{n!n^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!n^2}, \\
\sum_{n=1}^{\infty} \frac{\log(n+1)}{n^4} + \gamma\zeta(4) - 2\zeta(5) + \zeta(2)\zeta(3) - \frac{2}{3}\zeta(3) + \frac{1}{3}\zeta(2) - \frac{1}{2} &= \\
\frac{1}{6} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^3}{n!n^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n H_n^{(2)}}{n!n^2} + \frac{1}{3} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(3)}}{n!n^2}.
\end{aligned}$$

#### 5.4 Link with the Ramanujan summation

The function  $F_k$  has strong connections with the Ramanujan summation (cf. [3], [4]).

If  $a \in \mathcal{E}$ , then the series  $\sum_{n \geq 1} a(n)$  may be written

$$\sum_{n \geq 1} a(n) = \sum_{n \geq 1} \int_0^{+\infty} e^{-nt} \hat{a}(t) dt,$$

and a formal permutation of  $\sum_{n \geq 1}$  and  $\int_0^{+\infty}$  would lead us to write

$$\sum_{n \geq 1} a(n) = \int_0^{+\infty} \frac{1}{1 - e^{-t}} e^{-t} \hat{a}(t) dt.$$

However, this last integral may be divergent at 0. Nevertheless we can renormalize it by removing the singularity at zero. This may be done merely by subtracting the polar part  $\frac{1}{t}$  of  $\frac{1}{1 - e^{-t}}$ . From Theorem 1, we know that

$$\int_0^{+\infty} \left( \frac{1}{1 - e^{-t}} - \frac{1}{t} \right) e^{-t} \hat{a}(t) dt = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \int_0^{+\infty} e^{-t} (1 - e^{-t})^{n-1} \hat{a}(t) dt = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} D(a)(n).$$

This justifies the following definition.

**Definition 11.** Let  $a$  be a function in  $\mathcal{E} = \mathcal{L}(E)$ . The *Ramanujan sum* of the series  $\sum_{n \geq 1} a(n)$  is defined by

$$\sum_{n \geq 1}^{\mathcal{R}} a(n) = \int_0^{+\infty} \left( \frac{1}{1 - e^{-t}} - \frac{1}{t} \right) e^{-t} \hat{a}(t) dt = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} D(a)(n). \quad (29)$$

**Lemma 3.** Let  $a$  and  $b$  in  $\mathcal{E}$ . Then

$$\sum_{n \geq 1}^{\mathcal{R}} (a \bowtie b)(n) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} D(a)(n) D(b)(n). \quad (30)$$

*Proof.* This results directly from (12) and (29).  $\square$

**Theorem 11.** *for all  $s \in \mathbb{C}$  with  $\Re(s) \geq 1$ , one has*

$$F_0(s) = \sum_{n \geq 1}^{\mathcal{R}} \frac{1}{n^s} \quad \text{and} \quad F_k(s) = \sum_{n \geq 1}^{\mathcal{R}} \left( \left( \frac{1}{x} \right)^{\bowtie k} \bowtie \frac{1}{x^s} \right) (n) \quad \text{for } k \geq 1. \quad (31)$$

*Proof.* By (24) and (30), taking into account the invariance of  $\frac{1}{x}$  by  $D$ , one may write

$$\begin{aligned} \sum_{n \geq 1}^{\mathcal{R}} \left( \left( \frac{1}{x} \right)^{\bowtie k} \bowtie \frac{1}{x^s} \right) (n) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} D \left( \left( \frac{1}{x} \right)^{\bowtie k} \right) (n) D \left( \frac{1}{x^s} \right) (n) \\ &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \left( \frac{1}{x} \right)^k (n) D \left( \frac{1}{x^s} \right) (n) \\ &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^k} D \left( \frac{1}{x^s} \right) (n) = F_k(s). \end{aligned}$$

$\square$

In particular, by (14), one deduces from (31) the following identity.

**Corollary 3.** *For each natural number  $k$ ,*

$$F_k(1) = \vartheta(k+1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{1}{n^{k+1}} = \sum_{n \geq 1}^{\mathcal{R}} \frac{P_k(H_n, H_n^{(2)}, \dots, H_n^{(k)})}{n}. \quad (32)$$

**Example 10.**

$$\begin{aligned} \vartheta(1) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} = \sum_{n \geq 1}^{\mathcal{R}} \frac{1}{n} = \gamma, \\ \vartheta(2) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^2} = \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n}, \\ \vartheta(3) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^3} = \frac{1}{2} \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n^2}{n} + \frac{1}{2} \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n^{(2)}}{n}. \end{aligned}$$

**Remark 8.** Comparing (32) with

$$F_0(k+1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} P_k(H_n, H_n^{(2)}, \dots, H_n^{(k)}),$$

one may observe a kind of reciprocity between  $F_k(1)$  and  $F_0(k+1)$ . This results from the fact that  $D = D^{-1}$ .

**Remark 9.** In the case  $q = 1$ , (27) is meaningless since both the series  $\sum_{n \geq 1} \frac{\log(n+1)}{n}$  and  $\sum_{n \geq 1} \frac{H_n}{n}$  diverge. However, since

$$\log(x+1) - (\psi(x+1) + \gamma) = \int_0^{+\infty} (e^{-xu} - 1) \left( \frac{1}{1-e^{-u}} - \frac{1}{u} \right) e^{-u} du,$$

it follows that

$$\left( \frac{\widehat{\log(x+1)}}{x} - \frac{\widehat{\psi(x+1) + \gamma}}{x} \right) (t) = \int_t^{+\infty} \left( \frac{1}{1-e^{-u}} - \frac{1}{u} \right) e^{-u} du,$$

and then one may easily deduce from (29) the relation

$$\sum_{n \geq 1}^{\mathcal{R}} \frac{\log(n+1)}{n} - \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n} = -\frac{\gamma^2}{2},$$

which may be rewritten in the following form (cf. Example 10):

$$\sum_{n \geq 1}^{\mathcal{R}} \frac{\log(n+1)}{n} = \vartheta(2) - \frac{1}{2} \vartheta(1)^2.$$

## 5.5 Link with the Arakawa-Kaneko zeta function

For  $\Re(s) \geq 1$  and  $k \geq 1$ , one can define in an algebraic fashion the function  $\xi_k$  by

$$\xi_k(s) = \sum_{n=1}^{\infty} D \left( \left( \frac{1}{x} \right)^{\boxtimes k} \boxtimes \frac{1}{x^s} \right) (n) = \sum_{n=1}^{\infty} \frac{1}{n^k} D \left( \frac{1}{x^s} \right) (n). \quad (33)$$

In particular, for all natural numbers  $m$ , one has (cf. [8], Corollary 1)

$$\xi_k(m+1) = \sum_{n=1}^{\infty} \frac{1}{n^k} D \left( \frac{1}{x^{m+1}} \right) (n) = \sum_{n=1}^{\infty} \frac{P_m(H_n, H_n^{(2)}, \dots, H_n^{(m)})}{n^{k+1}}.$$

Since  $D \left( \frac{1}{x^s} \right) = \mathcal{L} \left( \frac{\Lambda^{s-1}}{\Gamma(s)} \right)$ , one may also rewrite (33) as

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \text{Li}_k(e^{-u}) (\Lambda(u))^{s-1} du,$$

and the change of variables  $t = \Lambda(u)$  leads to the integral representation

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} \text{Li}_k(1-e^{-t}) dt,$$

which is the analogue of (19) (with  $\text{Li}_k$  in place of  $f_k$ ) and also the original definition of the Arakawa-Kaneko zeta function (cf. [1], [8]).

Thus, taking into account the facts that  $\xi_k(1) = \zeta(k+1)$  and  $\text{Li}_1(1 - e^{-t}) = t$ , and following the same process as in the proof of Theorem 8, one obtains the following analogue of (20):

$$\xi_{k+1}(s) = \zeta(k+1)\zeta(s) + \sum_{j=1}^{k-1} (-1)^j \zeta(k+1-j) Z_j(s) + (-1)^k \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} T^k(t) dt. \quad (34)$$

In particular, in the simplest case  $k = 1$ , since

$$T(t) = \int_t^{+\infty} \frac{e^{-u}}{1-e^{-u}} u du = \sum_{m>0} \int_t^{+\infty} e^{-mu} u du = \sum_{m>0} \frac{e^{-tm}}{m} t + \sum_{m>0} \frac{e^{-tm}}{m^2},$$

(34) again gives the formula

$$\xi_2(s) = \zeta(2)\zeta(s) - s \sum_{n>m>0} \frac{1}{n^{s+1}} \frac{1}{m} - \sum_{n>m>0} \frac{1}{n^s} \frac{1}{m^2}$$

already obtained by Arakawa and Kaneko (cf. [1] Theorem 6 (ii)).

## 6 Conclusion

Most of the general results given for the modified zeta function  $F_k$ , especially Theorem 7, Theorem 8, and Theorem 9, also apply (with minor adaptations) to a wide class of functions including the Arakawa-Kaneko zeta function  $\xi_k$ , specifically to the class of functions represented by normalized Mellin transforms of type

$$F_{k,\omega}(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} f_{k,\omega}(1-e^{-t}) dt$$

with  $\omega = (\omega_n)_{n \geq 1}$  and  $f_{k,\omega}(z) = \sum_{n=1}^{\infty} \frac{\omega_n}{n^k} z^n$ . In particular, under the assumption that

$\frac{|\omega_n|}{n^k} = O\left(\frac{1}{n}\right)$ , we have for positive integers  $m$  the nice formula

$$F_{k,\omega}(m+1) = \sum_{n=1}^{\infty} \frac{\omega_n}{n^k} D\left(\frac{1}{x^{m+1}}\right)(n) = \sum_{n=1}^{\infty} \omega_n \frac{P_m(H_n, H_n^{(2)}, \dots, H_n^{(m)})}{n^{k+1}},$$

which extends (25). However, this formula is more theoretical than practical because of the fast increase in the size of polynomials  $P_m$ : the number of monomials in  $P_m$  is equal to the number  $p(m)$  of partitions of  $m$ , as shown by the explicit expression of the  $m$ th modified Bell polynomial.

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