A new class of identities involving Cauchy numbers, harmonic numbers and zeta values

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Article to appear in *The Ramanujan Journal*

Abstract

Improving an old idea of Hermite, we associate to each natural number k a modified zeta function of order k. The evaluation of the values of these functions F_k at positive integers reveals a wide class of identities linking Cauchy numbers, harmonic numbers and zeta values.

Mathematical Subject Classification (2000): 11B83, 11M41, 33B15, 40G99.

Keywords: Cauchy numbers, Bell polynomials, Harmonic numbers, Laplace-Borel transform, Mellin transform, Zeta values, Ramanujan summation, Hermite's formula.

1 Introduction

It has been well known since the second-half of the 19th century that the Riemann zeta function may be represented by the (normalized) Mellin transform (cf. [14])

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}}\, dt \quad \text{for } \Re(s) > 1\,,$$

and from late works of Hermite (cf. [11]) that one has also

$$\zeta(s) - \frac{1}{s-1} = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} \left(\sum_{n=1}^{\infty} \frac{\lambda_n}{n!} (1-e^{-t})^n \right) dt \quad \text{for } \Re(s) \geq 1\,,$$

where $\lambda_1 = \dfrac{1}{2}$ and $\lambda_{n+1} = \displaystyle\int_0^1 x(1-x)\cdots(n-x)\,dx$ are the (non-alternating) Cauchy numbers[1].

Improving Hermite's idea, one may, more generally, consider Mellin transforms of type

$$F(s) = \frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\frac{e^{-t}}{1-e^{-t}} f(1-e^{-t})\,dt \qquad \text{with } f(z) = \sum_{n=1}^{\infty} \omega_n \frac{z^n}{n^k}$$

for suitable sequences $(\omega_n)_{n\geq 1}$ of rational numbers. The simplest interesting case $\omega_n = 1$ corresponds to the *Arakawa-Kaneko zeta function* and has been studied extensively in [8]. In this article, we investigate the case $\omega_n = \dfrac{\lambda_n}{n!}$, *i.e.*, we study the function

$$F_k(s) = \frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\frac{e^{-t}}{1-e^{-t}} f_k(1-e^{-t})\,dt \qquad \text{with } f_k(z) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!}\frac{z^n}{n^k}\ (k=0,1,2,\dots),$$

which is *a priori* defined in the half-plane $\Re(s) \geq 1$ but analytically continues in the whole complex s-plane (Theorem 7). We call this function F_k the *modified zeta function of order* k. An evaluation by two different methods of the values of F_k at positive integers q leads to a new class of identities linking Cauchy numbers, harmonic numbers and zeta values. In the case $k = 0$, *Hermite's formula* for ζ (cf. [7]) is regained, *i.e.*,

$$F_0(q) = \zeta(q) - \frac{1}{q-1} = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} P_{q-1}(H_n^{(1)}, H_n^{(2)}, \dots, H_n^{(q-1)})\,,$$

where the polynomials P_m are the *modified Bell polynomials* defined by the generating function

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{z^k}{k}\right) = \sum_{m=0}^{\infty} P_m(x_1,\cdots,x_m)\,z^m\,,$$

evaluated at harmonic numbers $H_n^{(m)} = \displaystyle\sum_{j=1}^{n}\frac{1}{j^m}$. In the simplest higher case $k=1$, this extension of Hermite's formula leads to the following new relation (Theorem 10):

$$F_1(q) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^2} P_{q-1}(H_n, H_n^{(2)}, \dots, H_n^{(q-1)}) =$$
$$\sum_{n=1}^{\infty} \frac{\log(n+1)}{n^q} + \gamma\zeta(q) + \zeta(q+1) - \sum_{n=1}^{\infty}\frac{H_n}{n^q} - \sum_{k=1}^{q-1}\frac{1}{k}\sum_{n=1}^{\infty}\frac{1}{(n+1)^k n^{q-k}}\,,$$

where $H_n = H_n^{(1)}$, and $\gamma = \lim_{n\to\infty}(H_n - \log n)$ is the Euler-Mascheroni constant.

[1] The sequence of numbers $\frac{\lambda_n}{n!}$ appeared for the first time in a letter of James Gregory dated back to 1670 (cf. *The correspondence of Isaac Newton*, vol. 1, p. 46). For this reason, they are sometimes called *Gregory coefficients*.

For example, for $q = 2$, since $P_1(H_n) = H_n$ and $\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3)$ (cf. [6], [7]), then the previous relation may be written

$$F_1(2) = \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!n^2} = \sum_{n=1}^{\infty} \frac{\log(n+1)}{n^2} + \gamma\zeta(2) - \zeta(3) - 1\,,$$

and this generalizes the known formula

$$F_0(2) = \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!n} = \zeta(2) - 1\,.$$

The function F_k also has an interesting interpretation in terms of Ramanujan summation (cf. [3]) as underscored by Theorem 11. In particular, one shows the identity

$$F_k(1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{1}{n^{k+1}} = \sum_{n\geq 1}^{\mathcal{R}} \frac{P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{n}\,,$$

where, in the right member, $\sum_{n\geq 1}^{\mathcal{R}}$ denotes the sum (in the sense of Ramanujan) of the divergent series. This raises a kind of reciprocity between $F_k(1)$ and $F_0(k+1)$.

2 Preliminaries

2.1 The non-alternating Cauchy numbers

Definition 1. The *non-alternating Cauchy numbers* (cf. [7], [12]) are the sequence of (positive) rational numbers $(\lambda_n)_{n\geq 1}$ defined by the exponential generating function

$$\frac{z}{\log(1-z)} + 1 = \sum_{n\geq 1} \frac{\lambda_n}{n!} z^n\,. \tag{1}$$

Dividing by z and setting $z = 1 - e^{-t}$ and $t > 0$, this relation may be rewritten

$$\frac{1}{1-e^{-t}} - \frac{1}{t} = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} (1 - e^{-t})^{n-1}\,. \tag{2}$$

From (1), one may easily deduce the following recursive relation

$$\sum_{j=1}^{n} \frac{\lambda_j}{j!(n-j+1)} - \frac{1}{n+1} = 0 \quad \text{for } n \geq 1.$$

Example 1. The first non-alternating Cauchy numbers are

$$\lambda_1 = \frac{1}{2}\,, \lambda_2 = \frac{1}{6}\,, \lambda_3 = \frac{1}{4}\,, \lambda_4 = \frac{19}{30}\,, \lambda_5 = \frac{9}{4}\,.$$

2.2 The modified Bell polynomials evaluated at harmonic numbers

Definition 2. The *modified Bell polynomials* (cf. [5], [7], [10]) are the polynomials P_m defined for all natural numbers m by $P_0 = 1$ and the generating function

$$\exp\left(\sum_{k\geq 1} x_k \frac{z^k}{k}\right) = 1 + \sum_{m\geq 1} P_m(x_1, ..., x_m)\, z^m\,. \tag{3}$$

The general explicit expression for P_m is

$$P_m(x_1, ..., x_m) = \sum_{k_1+2k_2+3k_3+\dots=m} \frac{1}{k_1!k_2!k_3!\dots} \left(\frac{x_1}{1}\right)^{k_1} \left(\frac{x_2}{2}\right)^{k_2} \left(\frac{x_3}{3}\right)^{k_3} \dots$$

One may also compute recursively the polynomials P_m by means of the following relation

$$mP_m(x_1, \dots, x_m) = \sum_{k=1}^{m} x_k\, P_{m-k}(x_1, \dots, x_{m-k}) \quad (m \geq 1)\,.$$

Proposition 1. For all natural numbers m, and each integer $n \geq 1$,

$$\int_0^{+\infty} e^{-t}(1-e^{-t})^{n-1} \frac{t^m}{m!} dt = \frac{P_m(H_n, \dots, H_n^{(m)})}{n}\,, \tag{4}$$

with

$$H_n^{(m)} = \sum_{j=1}^{n} \frac{1}{j^m} \quad \text{and} \quad H_n = H_n^{(1)}\,.$$

Proof. One starts from the classical Euler relation (cf. [14])

$$\mathrm{B}(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\, du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,,$$

and substitute $u = e^{-t}$, $a = 1 - z$, and $b = n + 1$; then one obtains

$$\int_0^{+\infty} e^{-t}(1-e^{-t})^n e^{tz} dt = \frac{n!}{(1-z)(2-z)\dots(n+1-z)}\,.$$

Moreover, one has

$$\begin{aligned}
\frac{n!}{(1-z)(2-z)\dots(n+1-z)} &= \frac{n!}{(n+1)!} \times \prod_{j=0}^{n} (1 - \frac{z}{j+1})^{-1} \\
&= \frac{1}{(n+1)} \times \exp(-\sum_{j=0}^{n} \log(1 - \frac{z}{j+1})) \\
&= \frac{1}{(n+1)} \times \exp(\sum_{j=0}^{n}\sum_{k=1}^{\infty} \frac{z^k}{k(j+1)^k}) \\
&= \frac{1}{(n+1)} \exp(\sum_{k=1}^{\infty} H_{n+1}^{(k)} \frac{z^k}{k}) \\
&= \sum_{m=0}^{\infty} \frac{P_m(H_{n+1}^{(1)}, \dots, H_{n+1}^{(m)})}{n+1} z^m \quad \text{(by (3))}.
\end{aligned}$$

Thus (4) results by identification of the term in z^m. $\square$

Example 2. For small values of m, one has

$$P_1(H_n) = H_n,\ P_2(H_n, H_n^{(2)}) = \frac{(H_n)^2}{2} + \frac{H_n^{(2)}}{2},$$

$$P_3(H_n, H_n^{(2)}, H_n^{(3)}) = \frac{(H_n)^3}{6} + \frac{H_n H_n^{(2)}}{2} + \frac{H_n^{(3)}}{3}.$$

2.3 The Laplace-Borel transformation

We consider the vector space E of complex-valued functions $f \in \mathcal{C}^1(]0, +\infty[)$ such that

$$\text{for all } \varepsilon > 0, \text{ there exists } C_\varepsilon > 0 \text{ such that } |f(t)| \le C_\varepsilon e^{\varepsilon t} \text{ for all } t \in]0, +\infty[\ .$$

In particular, a function $f \in E$ satisfies the following two properties:

a) for all x with $\Re(x) > 0$, $t \mapsto e^{-xt} f(t)$ is integrable on $]0, +\infty[$,

b) for all β with $0 < \beta < 1$, $t \mapsto |f(t)| \frac{1}{t^\beta}$ is integrable on $]0, 1[$.

We recall now some basic properties (cf. [13]) of the Laplace transformation in this frame which are appropriate for our purpose.

Definition 3. Let f be a function in E. The *Laplace transform* $\mathcal{L}(f)$ of f is defined by

$$\mathcal{L}(f)(x) = \int_0^{+\infty} e^{-xt} f(t)\, dt \quad \text{for } \Re(x) > 0\,.$$

Proposition 2 (cf. [13])**.** Let $\mathcal{E} = \mathcal{L}(E)$ be the image of E under $\mathcal{L}$. If a is a function in $\mathcal{E}$, then

a) a is an analytic function of x in the half-plane $\Re(x) > 0$,

b) $a(x) \to 0$ when $\Re(x) \to +\infty$,

c) $\mathcal{L} : E \to \mathcal{E}$ is an isomorphism.

Definition 4. Let $a \in \mathcal{E}$. The *Borel transform* of a is the unique function $\widehat{a} \in E$ such that $a = \mathcal{L}(\widehat{a})$. One has the two reciprocal formulas

$$\widehat{a}(t) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} e^{zt} a(z)\, dz \quad \text{for all } c > 0 \text{ and } t > 0\,,$$

and

$$a(x) = \int_0^{+\infty} e^{-xt} \widehat{a}(t)\, dt \quad \text{for } \Re(x) > 0\,.$$

Definition 5. Let f and g be two functions in E. The *convolution product* $f * g$ of f and g is the function defined for all $t > 0$ by

$$(f * g)(t) = \int_0^t f(u)g(t-u)\,du\,.$$

Proposition 3 (cf. [13])**.** If $f \in E$ and $g \in E$, then $f * g \in E$ and

$$\mathcal{L}(f * g) = \mathcal{L}(f)\,\mathcal{L}(g)\,. \tag{5}$$

Hence, if $a \in \mathcal{E}$ and $b \in \mathcal{E}$ then $ab \in \mathcal{E}$ since $ab = \mathcal{L}(\widehat{a} * \widehat{b})$.

Theorem 1. *Let a be a function in $\mathcal{E}$. Then the series*

$$\sum_{n\geq 1} \frac{\lambda_n}{n!} \int_0^{+\infty} e^{-t}(1-e^{-t})^{n-1}\widehat{a}(t)\,dt$$

converges and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \int_0^{+\infty} e^{-t}(1-e^{-t})^{n-1}\widehat{a}(t)dt = \int_0^{+\infty} (\frac{1}{1-e^{-t}} - \frac{1}{t})e^{-t}\widehat{a}(t)\,dt\,. \tag{6}$$

Proof. By (2)

$$\int_0^{+\infty} (\frac{1}{1-e^{-t}} - \frac{1}{t})e^{-t}\widehat{a}(t)dt = \int_0^{+\infty} \sum_{n=1}^{\infty} \frac{\lambda_n}{n!}(1-e^{-t})^{n-1}e^{-t}\widehat{a}(t)\,dt\,.$$

In the right member, the order of $\int_0^{+\infty}$ and $\sum_{n=1}^{\infty}$ may be interchanged since

$$\begin{aligned}\int_0^{+\infty} \sum_{n=1}^{\infty} \left|\frac{\lambda_n}{n!}(1-e^{-t})^{n-1}e^{-t}\widehat{a}(t)\right| dt &= \int_0^{+\infty} \sum_{n=1}^{\infty} \frac{\lambda_n}{n!}(1-e^{-t})^{n-1}e^{-t}\,|\widehat{a}(t)|\,dt \\ &= \int_0^{+\infty} (\frac{1}{1-e^{-t}} - \frac{1}{t})e^{-t}\,|\widehat{a}(t)|\,dt\,,\end{aligned}$$

and the convergence of this last integral follows from the assumption that $a \in \mathcal{E}$. $\square$

Example 3. Let $a(x) = \dfrac{1}{x^s}$ with $\Re(s) \geq 1$. Then $a \in \mathcal{E}$ and $\widehat{a}(t) = \dfrac{t^{s-1}}{\Gamma(s)}$. Hence

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \int_0^{+\infty} e^{-t}(1-e^{-t})^{n-1}\frac{t^{s-1}}{\Gamma(s)}dt &= \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-t}(\frac{1}{1-e^{-t}} - \frac{1}{t})t^{s-1}dt \\ &= \begin{cases} \gamma & \text{if } s = 1, \\ \zeta(s) - \dfrac{1}{s-1} & \text{if } s \neq 1 \end{cases}\end{aligned}$$

where γ refers to the Euler constant. In particular, since

$$\int_0^{+\infty} e^{-t}(1-e^{-t})^{n-1}dt = \frac{1}{n} \quad \text{for each integer } n \geq 1\,,$$

then

$$\gamma = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!}\frac{1}{n}\,.$$

3 The operator D

Proposition 4. If $a \in \mathcal{E}$, then the integral

$$\int_0^{+\infty} e^{-t}(1-e^{-t})^{x-1}\widehat{a}(t)dt$$

converges for all x with $\Re(x) > 0$.

Proof. If $a \in \mathcal{E}$ and $\Re(x) > 0$, we may write for $t \in]0, +\infty[$,

$$\left| e^{-t}(1-e^{-t})^{x-1}\widehat{a}(t) \right| \le e^{-t} e^{(1-\Re(x))(-\log(1-e^{-t}))} \left|\widehat{a}(t)\right| .$$

The convergence when $t \to +\infty$ results from the inequality

$$e^{-t} e^{(1-\Re(x))(-\log(1-e^{-t}))} \left|\widehat{a}(t)\right| \le \frac{e^{-t}}{1-e^{-t}} \left|\widehat{a}(t)\right| \le 2e^{-t} \left|\widehat{a}(t)\right| \quad (\text{for } t \ge \log 2).$$

The convergence when $t \to 0$ results from the inequality

$$e^{(1-\Re(x))(-\log(1-e^{-t}))} \le \begin{cases} 1 & \text{if } \Re(x) \ge 1, \\ \frac{1}{(1-e^{-t})^{(1-\Re(x))}} & \text{if } 0 < \Re(x) < 1 \end{cases}$$

since the function $t \mapsto e^{-t}\left|\widehat{a}(t)\right| \dfrac{1}{(1-e^{-t})^{\beta}}$ is integrable at 0 for $0 < \beta < 1$ by the definition of E (note that $(1-e^{-t})^{-\beta} \le (kt)^{-1}$ for small enough t). $\square$

Definition 6. Let a be a function in $\mathcal{E}$. We call $D(a)$ the function defined for all x with $\Re(x) > 0$ by

$$D(a)(x) = \int_0^{+\infty} e^{-t}(1-e^{-t})^{x-1}\widehat{a}(t)dt \,. \tag{7}$$

Remark 1. a) By Theorem 1, the series $\sum_{n\ge1} \frac{\lambda_n}{n!} D(a)(n)$ converges and its sum is given by formula (6).

b) The values of $D(a)$ at positive integers may be computed directly without recourse to $\widehat{a}$. The development of $(1-e^{-t})^n$ by the binomial theorem gives

$$D(a)(n+1) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a(k+1) \quad \text{for all integer } n \ge 0. \tag{8}$$

Definition 7. We call Λ the C^1-diffeomorphism of $\mathbb{R}_+$ defined by $\Lambda(u) = -\log(1-e^{-u})$. In particular, it is important to note that Λ is involutive:

$$\Lambda^{-1} = \Lambda \,.$$

Theorem 2. *Let a be a function in $\mathcal{E}$. Then the function $D(a) \in \mathcal{E}$ and, moreover, verifies the relation*

$$\widehat{D(a)} = \widehat{a}(\Lambda)\,, \tag{9}$$

where $\widehat{a}(\Lambda)$ denotes $\widehat{a} \circ \Lambda$.

Proof. The change of variables $t = \Lambda(u)$ in (7) gives

$$D(a)(x) = \int_0^{+\infty} e^{-xu}\widehat{a}(\Lambda(u))\,du \quad \text{for } \Re(x) > 0.$$

Thus, $D(a) = \mathcal{L}(\widehat{a}(\Lambda))$. It remains to prove that $D(a) \in \mathcal{E}$. One has only to check that the function $\widehat{a}(\Lambda)$ is in E. This function being in $\mathcal{C}^1(]0,+\infty[)$, it suffices to show that for all $\varepsilon > 0$, the function $u \mapsto e^{-\varepsilon u}\,|\widehat{a}(-\log(1-e^{-u}))|$ is bounded on $]0,+\infty[$. This results from the existence of $C_\varepsilon > 0$ such that

$$\left|\widehat{a}(-\log(1-e^{-u}))\right| \le C_\varepsilon(1-e^{-u})^\varepsilon \ \text{ for all } u \in\,]0,+\infty[\ .$$

□

Example 4. Let $a(x) = \dfrac{1}{x^s}$ with $\Re(s) \ge 1$. Then $\widehat{a}(t) = \dfrac{t^{s-1}}{\Gamma(s)}$. Thus, by (9),

$$D(\frac{1}{x^s}) = \mathcal{L}\left(\frac{\Lambda^{s-1}}{\Gamma(s)}\right)\,, \tag{10}$$

and if $s = m+1$ with m a natural number and $n \ge 1$, then by (4),

$$D(\frac{1}{x^{m+1}})(n) = \frac{P_m(H_n,\ldots,H_n^{(m)})}{n}\,. \tag{11}$$

Remark 2. Theorem 2 may be summarized in the following diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{D} & \mathcal{E} \\ \big\downarrow \mathcal{L}^{-1} & & \big\uparrow \mathcal{L} \\ E & \xrightarrow{\Lambda^\star} & E \end{array}$$

where $\Lambda^\star(\widehat{a}) = \widehat{a}(\Lambda)$. The algebraic properties of D are summed up in the following theorem.

Theorem 3. *The operator D is an automorphism of $\mathcal{E}$ which verifies $D = D^{-1}$ and lets the function $x \mapsto \dfrac{1}{x}$ invariant.*

Proof. We can write $D = \mathcal{L}\Lambda^\star\mathcal{L}^{-1}$ and $\Lambda^\star$ is an automorphism of E which verifies $\Lambda^\star = (\Lambda^\star)^{-1}$ since $\Lambda = \Lambda^{-1}$. Furthermore,

$$D(\frac{1}{x}) = \mathcal{L}(1) = \frac{1}{x}\,.$$

□

4 The harmonic product

Our aim is to define the harmonic product of two functions a and b in $\mathcal{E}$ as being the unique function f of $\mathcal{E}$ such that

$$D(a)\,(x).D(b)\,(x) = D(f)\,(x)\,.$$

Thus, we have to establish that such a function exists and is unique. In order to do this, we introduce first a Λ-convolution product of two functions in E.

4.1 The Λ-convolution product

Proposition 5. If a and b are in $\mathcal{E}$, then $\widehat{a}(\Lambda) * \widehat{b}(\Lambda) \in E$.

Proof. From the definition of the convolution product, one may write

$$\left(\widehat{a}(\Lambda) * (\widehat{b}(\Lambda)\right)(t) = \int_0^t \widehat{a}(\Lambda(u))\widehat{b}(\Lambda(t-u))du\,.$$

Now, for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ and $D_\varepsilon > 0$ such that

$$\begin{aligned} \left|\widehat{a}(-\log(1-e^{-u}))\right| &\leq C_\varepsilon(1-e^{-u})^\varepsilon \text{ and} \\ \left|\widehat{b}(-\log(1-e^{-(t-u)}))\right| &\leq D_\varepsilon(1-e^{-(t-u)})^\varepsilon \text{ for all } u \in\,]0,+\infty[\,. \end{aligned}$$

It follows that

$$\left|(\widehat{a}(\Lambda) * \widehat{b}(\Lambda))(t)\right| \leq C_\varepsilon D_\varepsilon \int_0^t (1-e^{-u})^\varepsilon(1-e^{-(t-u)})^\varepsilon du\,.$$

One has also

$$\begin{aligned} \int_0^t (1-e^{-u})^\varepsilon(1-e^{-(t-u)})^\varepsilon du &= \left(1-e^{-t}\right)^{1+2\varepsilon} \int_0^1 u^\varepsilon\,(1-u)^\varepsilon \frac{1}{(1-(1-e^{-t})u)^{\varepsilon+1}}\,du \\ &\leq \left(1-e^{-t}\right)^{1+2\varepsilon} \int_0^1 \frac{1}{(1-(1-e^{-t})u)^{\varepsilon+1}}\,du \leq \left(1-e^{-t}\right)^{1+2\varepsilon} \frac{e^{t\varepsilon}-1}{(1-e^{-t})\varepsilon} \\ &\leq \left(1-e^{-t}\right)^{2\varepsilon} \frac{e^{t\varepsilon}-1}{\varepsilon} \leq \frac{e^{t\varepsilon}}{\varepsilon}. \end{aligned}$$

Hence, $\left|(\widehat{a}(\Lambda) * \widehat{b}(\Lambda))(t)\right| \leq C_\varepsilon D_\varepsilon \frac{e^{t\varepsilon}}{\varepsilon}$, which proves that this function belongs to E as required. $\square$

Definition 8. Let a and b be two functions in $\mathcal{E}$. The Λ-*convolution product* $\widehat{a} \circledast \widehat{b}$ of $\widehat{a}$ and $\widehat{b}$ is defined by

$$\widehat{a} \circledast \widehat{b} = \Lambda^\star(\Lambda^\star(\widehat{a}) * \Lambda^\star(\widehat{b}))\,,$$

or equivalently (since $\Lambda^\star = (\Lambda^\star)^{-1}$)

$$(\widehat{a} \circledast \widehat{b})(\Lambda) = \widehat{a}(\Lambda) * \widehat{b}(\Lambda)\,.$$

Remark 3. The Λ-convolution product inherits the algebraic properties of the ordinary convolution product, *i.e.*, bilinearity, commutativity, and associativity.

4.2 The harmonic product

Definition 9. Let a and b two functions in $\mathcal{E}$. The *harmonic product* $a \bowtie b$ of a and b is defined by

$$a \bowtie b = \mathcal{L}(\widehat{a} \circledast \widehat{b}) \in \mathcal{E}\,.$$

This construction may be summarized in the following diagram:

$$\begin{array}{ccccc} (a,b) & \longrightarrow & (\widehat{a},\widehat{b}) & \longrightarrow & (\widehat{a}(\Lambda),\widehat{b}(\Lambda)) \\ \downarrow & & \downarrow & & \downarrow \\ a \bowtie b & \longleftarrow & \widehat{a} \circledast \widehat{b} & \longleftarrow & \widehat{a}(\Lambda) * \widehat{b}(\Lambda) \end{array}$$

Remark 4. The harmonic product inherits the properties of the Λ-convolution product: it is bilinear, commutative and associative.

Theorem 4. *Let a and b be in $\mathcal{E}$. Then,*

$$D(a \bowtie b) = D(a)\,D(b)\,, \tag{12}$$

and

$$D(ab) = D(a) \bowtie D(b)\,. \tag{13}$$

Proof. One knows from Theorem 2 that

$$D = \mathcal{L}\Lambda^\star\mathcal{L}^{-1}\,.$$

Hence

$$D(a \bowtie b) = \mathcal{L}\Lambda^\star\mathcal{L}^{-1}(a \bowtie b) = \mathcal{L}\Lambda^\star(\widehat{a} \circledast \widehat{b}) = \mathcal{L}(\Lambda^\star(\widehat{a}) * \Lambda^\star(\widehat{b}))\,,$$

and it follows from (5) and (9) that

$$\mathcal{L}(\Lambda^\star(\widehat{a}) * \Lambda^\star(\widehat{b})) = \mathcal{L}(\Lambda^\star(\widehat{a}))\mathcal{L}(\Lambda^\star(\widehat{b})) = D(a)\,D(b)$$

which proves (12). Moreover, (12) enables us to write

$$D(D(a) \bowtie D(b)) = D^2(a)\,D^2(b) = ab \quad (\text{since } D = D^{-1}),$$

and so

$$D(a\,b) = D^2(D(a) \bowtie D(b)) = D(a) \bowtie D(b)$$

which proves (13). □

Remark 5. The values of $(a \bowtie b)(n)$ may be computed without recourse to $\widehat{a}$ and $\widehat{b}$. By elementary transformations, it can be shown that

$$(a \bowtie b)(n+1) = \int_0^{+\infty}\int_0^{+\infty}(e^{-t-s})(e^{-t}+e^{-s}-e^{-t}e^{-s})^n\widehat{a}(t)\widehat{b}(s)dtds\,.$$

Hence, if the numbers $C_n^{k,l}$ are defined by

$$(X+Y-XY)^n = \sum_{\substack{0\le k\le n\\ 0\le l\le n}} C_n^{k,l}\, X^k Y^l\,,$$

then one has the following explicit formula:

$$(a \bowtie b)(n+1) = \sum_{\substack{0\le k\le n\\ 0\le l\le n}} C_n^{k,l}\, a(k+1)b(l+1)\,,$$

which can be rewritten in the following equivalent form:

$$(a \bowtie b)(n+1) = \sum_{0\le l\le k\le n} (-1)^{k-l}\binom{n}{k}\binom{k}{l}\, a(k+1)b(n+1-l) \quad (n\ge 0)\,.$$

For small values of n, this enables one to compute

$$\begin{aligned}
&(a \bowtie b)(1) = a(1)b(1)\,,\\
&(a \bowtie b)(2) = a(2)b(1) + a(1)b(2) - a(2)b(2)\,,\\
&(a \bowtie b)(3) = a(3)b(1) + a(1)b(3) + 2a(2)b(2) - 2a(3)b(2) - 2a(2)b(3) + a(3)b(3)\,.
\end{aligned}$$

Theorem 5. *Let*

$$\left(\frac{1}{x}\right)^{\bowtie k} = \underbrace{\frac{1}{x} \bowtie \frac{1}{x} \bowtie \cdots \bowtie \frac{1}{x}}_{k} \quad (k=1,2,3,\cdots)\,,$$

where $\frac{1}{x}$ *denotes (improperly) the function* $x \mapsto \frac{1}{x}$. *Then, for all natural numbers* $m \ge 0$,

$$\left(\frac{1}{x}\right)^{\bowtie (m+1)} = D(\frac{1}{x^{m+1}})\,.$$

In particular, for all integers $n \ge 1$,

$$\left(\frac{1}{x}\right)^{\bowtie (m+1)}(n) = \frac{P_m(H_n,\ldots,H_n^{(m)})}{n}. \tag{14}$$

Proof. By (13) we have

$$D(\frac{1}{x^{m+1}}) = D(\underbrace{\frac{1}{x}\cdots\frac{1}{x}}_{m+1}) = \left(D(\frac{1}{x})\right)^{\bowtie (m+1)} = \left(\frac{1}{x}\right)^{\bowtie (m+1)} \text{ since } D(\frac{1}{x}) = \frac{1}{x}.$$

Thus, (14) results from (11). $\square$

4.3 The harmonic property

The following theorem explains the main reason why the harmonic product is called 'harmonic'.

Theorem 6. *Let $a \in \mathcal{E}$. Then*

$$\frac{1}{x} \bowtie a = \frac{A(x)}{x}\,,$$

where A denotes the function defined for $\Re(x) > 0$ by

$$A(x) = \int_0^{+\infty} \frac{e^{-xt} - 1}{e^{-t} - 1} e^{-t} \widehat{a}(t) dt\,.$$

In particular, for each integer $n \geq 1$,

$$\left(\frac{1}{x} \bowtie a\right)(n) = \frac{A(n)}{n} = \frac{1}{n}\left(\sum_{k=1}^{n} a(k)\right)\,. \tag{15}$$

Proof. By the definition of the harmonic product, one has

$$\frac{1}{x} \bowtie a = \mathcal{L}(1 \circledast \widehat{a})\,.$$

Now

$$(1 \circledast \widehat{a})(\Lambda(u)) = (1 * \widehat{a}(\Lambda))(u) = \int_0^u \widehat{a}(\Lambda(v)) dv = -\int_{+\infty}^{\Lambda(u)} \widehat{a}(t) \frac{e^{-t}}{1 - e^{-t}} dt$$

(by the change of variables $t = \Lambda(v)$). Hence,

$$(1 \circledast \widehat{a})(u) = \int_u^{+\infty} \widehat{a}(t) \frac{e^{-t}}{1 - e^{-t}} dt\,.$$

Thus, we have

$$\begin{aligned}
\frac{1}{x} \bowtie a &= \int_0^{+\infty} e^{-xu} \left(\int_u^{+\infty} \widehat{a}(t) \frac{e^{-t}}{1 - e^{-t}} dt\right) du \\
&= \int_0^{+\infty} \left(\int_0^t e^{-xu} du\right) \widehat{a}(t) \frac{e^{-t}}{1 - e^{-t}} dt \\
&= \frac{1}{x} \int_0^{+\infty} (1 - e^{-xt}) \widehat{a}(t) \frac{e^{-t}}{1 - e^{-t}} dt \\
&= \frac{A(x)}{x}\,.
\end{aligned}$$

Furthermore, for each integer $n \geq 1$, we have

$$A(n) = \int_0^{+\infty} \frac{e^{-nt} - 1}{e^{-t} - 1} e^{-t} \widehat{a}(t) dt = \sum_{k=1}^{n} a(k)\,.$$

$\square$

Example 5.

$$\frac{1}{x} \bowtie \frac{1}{x} = D(\frac{1}{x^2}) = \mathcal{L}(\Lambda) = \frac{H(x)}{x} \quad \text{with } H(x) = \psi(x+1) + \gamma,$$

ψ denoting the logarithmic derivative of Γ. In particular, for each integer $n \geq 1$,

$$\left(\frac{1}{x} \bowtie \frac{1}{x}\right)(n) = \frac{H(n)}{n} = \frac{H_n}{n}.$$

Example 6. For $\Re(s) \geq 1$,

$$\frac{1}{x} \bowtie \frac{1}{x^s} = \frac{H^{(s)}(x)}{x},$$

with

$$H^{(s)}(x) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{1 - e^{-xt}}{1 - e^{-t}} e^{-t} t^{s-1} \, dt.$$

For each integer $n \geq 1$,

$$\left(\frac{1}{x} \bowtie \frac{1}{x^s}\right)(n) = \frac{H^{(s)}(n)}{n} = \frac{H_n^{(s)}}{n} = \frac{1}{n}\left(\sum_{m=1}^{n} \frac{1}{m^s}\right).$$

From (15), by induction on k, we deduce the following important corollary.

Corollary 1. *For each integer $k \geq 2$,*

$$\left(\left(\frac{1}{x}\right)^{\bowtie k} \bowtie a\right)(n) = \frac{1}{n}\left(\sum_{n \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{a(n_k)}{n_1 \dots n_{k-1}}\right). \tag{16}$$

Example 7. Applying (16) with $a(x) = \frac{1}{x}$ (and $k = m$), we get

$$\left(\frac{1}{x}\right)^{\bowtie (m+1)}(n) = \frac{1}{n}\left(\sum_{n \geq n_1 \geq \cdots \geq n_m \geq 1} \frac{1}{n_1 \dots n_m}\right). \tag{17}$$

Hence, it follows from (14) and (17) that

$$P_m(H_n, H_n^{(2)}, \dots, H_n^{(m)}) = \sum_{n \geq n_1 \geq \cdots \geq n_m \geq 1} \frac{1}{n_1 \dots n_m}, \tag{18}$$

which is a nice reformulation of Dilcher's formula (cf. [2], [9]).

5 The modified zeta function F_k

5.1 Integral representation

Definition 10. For all $s \in \mathbb{C}$ with $\Re(s) \geq 1$ and each natural number k, the *modified zeta function of order k* is defined by

$$F_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1 - e^{-t}} f_k(1 - e^{-t}) \, dt \quad \text{with} \quad f_k(z) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{z^n}{n^k}. \tag{19}$$

Remark 6. By (2) and Example 3, one has $F_0(s) = \zeta(s) - \dfrac{1}{s-1}$.

The fact that F_k may be represented by a Mellin transform enables us to analytically continue this function outside its half-plane of definition by a standard analytic method (cf. [14] section 6.7).

Theorem 7. *The function F_k analytically continues in the whole complex plane as an entire function.*

Proof. The function $z \mapsto \dfrac{1}{\log(1-z)} + \dfrac{1}{z}$ being analytic in the disc $D(0,1)$ with a singularity at 1, we deduce from (1) that the radius of convergence of the series $\sum_{n=1}^{\infty} \dfrac{\lambda_n z^n}{n!}$ is equal to 1. Thus 1 is also the radius of convergence of the series $\sum_{n=1}^{\infty} \dfrac{\lambda_n z^n}{n! n^k}$ which defines an analytic function f_k in the disc $D(0,1)$. Hence, the function

$$g_k : t \mapsto f_k(1 - e^{-t})$$

is analytic for all $t \in \mathbb{C}$ such that $1 - e^{-t} \in D(0,1)$. Since $1 - e^0 = 0$, it follows that g_k is analytic in a neighbourhood of 0. Since $g_k(0) = 0$, the function $t \mapsto g_k(t)\dfrac{e^{-t}}{1-e^{-t}}$ is itself analytic in a neighbourhood of 0. It follows that its Mellin transform analytically continues in the complex plane with simple poles at negative integers which are all cancelled by the poles of Γ. $\square$

Theorem 8. *For all s with $\Re(s) > 1$ and each integer $k \geq 1$,*

$$F_k(s) = \vartheta(k)\zeta(s) + \sum_{j=1}^{k} (-1)^j \vartheta(k-j) Z_j(s) + (-1)^k \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} T^k \left(\frac{e^{-t}-1}{t} \right) dt \tag{20}$$

with

$$\vartheta(k) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{1}{n^k}, \tag{21}$$

$$Z_j(s) = \sum_{n > n_1 > n_2 > \cdots > n_j > 0} \frac{1}{n^s n_1 n_2 \ldots n_j}, \tag{22}$$

$$Tf\,(t) = \int_t^{+\infty} \frac{e^{-u}}{1-e^{-u}} f(u) du\,. \tag{23}$$

Proof. Formula (20) results from the integral representation (19) and the two following lemmas.

Lemma 1. For all $t > 0$,

$$f_k(1 - e^{-t}) = \sum_{j=0}^{k} (-1)^j \vartheta(k-j) \frac{\Lambda^j\,(t)}{j!} + (-1)^k T^k \left(\frac{e^{-t}-1}{t}\right),$$

where ϑ is defined by (21) and T is the operator defined by (23).

Proof. Let $g_k(t) = f_k(1-e^{-t})$. The function g_k verifies the recursive relation

$$g_k'(t) = e^{-t} f_k'(1-e^{-t}) = \frac{e^{-t}}{1-e^{-t}} f_{k-1}(1-e^{-t}) = \frac{e^{-t}}{1-e^{-t}} g_{k-1}(t)\,.$$

Thus

$$g_k(t) = \int_0^t \frac{e^{-u}}{1-e^{-u}} g_{k-1}(u)du = g_k(+\infty) - \int_t^{+\infty} \frac{e^{-u}}{1-e^{-u}} g_{k-1}(u)du$$

with

$$g_k(+\infty) = f_k(1) = \vartheta(k).$$

Thus, one has

$$g_k(t) = \vartheta(k) - \int_t^{+\infty} \frac{e^{-u}}{1-e^{-u}} g_{k-1}(u)du = \vartheta(k) - T(g_{k-1})\,,$$

and a repeated iteration k times of this relation gives

$$g_k(t) = \sum_{j=0}^{k-1} \vartheta(k-j)(-1)^j T^j\,(1) + (-1)^k T^k(g_0)\,.$$

Now, by (2),

$$g_0(t) = \sum_{n=1}^{\infty} \frac{\lambda_n (1-e^{-t})^n}{n!} = \frac{e^{-t}-1}{t} + 1\,,$$

and thus

$$T^k(g_0) = T^k(\frac{e^{-t}-1}{t}) + T^k(1)\,.$$

Hence

$$g_k(t) = \sum_{j=0}^{k-1} \vartheta(k-j)(-1)^j T^j\,(1) + (-1)^k T^k(1) + (-1)^k T^k(\frac{e^{-t}-1}{t})\,.$$

Since $\vartheta(0) = \sum_{n=1}^{\infty} \dfrac{\lambda_n}{n!} = 1$ (by (1) and a tauberian theorem), one deduces that

$$g_k(t) = \sum_{j=0}^{k} \vartheta(k-j)(-1)^j T^j\,(1) + (-1)^k T^k(\frac{e^{-t}-1}{t})\,,$$

and, now, it remains to prove that

$$\frac{\Lambda^j\,(t)}{j!} = T^j(1)\,,$$

which follows from the recursive relation

$$\frac{\Lambda^j\,(t)}{j!} = -\int_{+\infty}^t \frac{e^{-u}}{1-e^{-u}} \frac{\Lambda^{j-1}\,(u)}{(j-1)!} du = T\left(\frac{\Lambda^{j-1}}{(j-1)!}\right)\,.$$

$\square$

Lemma 2. Let $Z_j(s)$ be defined by (22). Then, for all $s \in \mathbb{C}$ with $\Re(s) > 1$,

$$Z_j(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} \frac{\Lambda^j\,(t)}{j!} dt\,.$$

Proof. From the recursive relation

$$\partial \frac{\Lambda^j\,(t)}{j!} = \frac{\Lambda^{j-1}\,(t)}{(j-1)!} \partial\Lambda(t) = -\frac{e^{-t}}{1-e^{-t}} \frac{\Lambda^{j-1}\,(t)}{(j-1)!} = -\sum_{m>0} e^{-mt} \frac{\Lambda^{j-1}\,(t)}{(j-1)!}\,,$$

and $\Lambda(t) = \sum_{n>0} \frac{e^{-nt}}{n}$, one may check by induction on j that

$$\frac{\Lambda^j\,(t)}{j!} = \sum_{n_1>n_2>\ldots>n_j>0} \frac{e^{-n_1 t}}{n_1} \frac{1}{n_2} \cdots \frac{1}{n_j}\,.$$

Furthermore, one has

$$\frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} e^{-Nt} \frac{e^{-t}}{1-e^{-t}} dt = \sum_{n>N} \frac{1}{n^s} \quad (\text{for } \Re(s) > 1)\,.$$

Hence

$$\frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} \frac{\Lambda^j\,(t)}{j!} dt = \sum_{n>n_1>n_2>\cdots>n_j>0} \frac{1}{n^s} \frac{1}{n_1} \frac{1}{n_2} \cdots \frac{1}{n_j} = Z_j(s)\,.$$

□

□

5.2 Values of F_k at integers

Theorem 9. *For all s in $\mathbb{C}$ with $\Re(s) \geq 1$ and each natural number k, then*

$$F_k(s) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^k} D\left(\frac{1}{x^s}\right)(n)\,. \tag{24}$$

In particular, for all natural numbers m,

$$F_k(m+1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{P_m(H_n, H_n^{(2)}, \ldots, H_n^{(m)})}{n^{k+1}}\,. \tag{25}$$

Proof. The change of variables $t = \Lambda(u)$ in (19) enables to write

$$F_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} f_k(e^{-u})(\Lambda(u))^{s-1} du\,.$$

Since $D(\frac{1}{x^s}) = \mathcal{L}\left(\frac{\Lambda^{s-1}}{\Gamma(s)}\right)$, we deduce (24) from this last expression of $F_k(s)$. Moreover, by (11), one also has $D(\frac{1}{x^{m+1}})(n) = \frac{P_m(H_n, \ldots, H_n^{(m)})}{n}$, which proves (25). □

Corollary 2. *Let $\vartheta(s)$ be the Dirichlet series defined for $\Re(s) > 0$ by*

$$\vartheta(s) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{1}{n^s} \,.$$

Then for each natural number k,

$$\vartheta(k+1) = F_k(1) \,. \tag{26}$$

Example 8.

$$\begin{aligned}
F_0(1) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} = \gamma = \vartheta(1) \,, \\
F_0(2) &= \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!n} = \zeta(2) - 1 \,, \\
F_0(3) &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^2}{n!n} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!n} = \zeta(3) - \frac{1}{2} \,, \\
F_1(1) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^2} = \vartheta(2) \,, \\
F_1(2) &= \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!n^2} \,, \\
F_1(3) &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^2}{n!n^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!n^2} \,.
\end{aligned}$$

5.3 Identities linking Cauchy numbers, harmonic numbers and zeta values

Theorem 10. *For all integers $q \geq 2$,*

$$\begin{aligned}
F_1(q) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^2} P_{q-1}(H_n, H_n^{(2)}, \ldots, H_n^{(q-1)}) = \\
\sum_{n=1}^{\infty} \frac{\log(n+1)}{n^q} + \gamma\zeta(q) + \zeta(q+1) - \sum_{n=1}^{\infty} \frac{H_n}{n^q} - \sum_{k=1}^{q-1} \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{(n+1)^k n^{q-k}} \,.
\end{aligned} \tag{27}$$

Proof. By (20) and (25), one may write

$$\begin{aligned}
F_k(q) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^{k+1}} P_{q-1}(H_n, H_n^{(2)}, \ldots, H_n^{(q-1)}) = \\
\vartheta(k)\zeta(q) + \sum_{j=1}^{k} (-1)^j \vartheta(k-j) Z_j(q) + (-1)^k \frac{1}{\Gamma(q)} \int_0^{+\infty} t^{q-1} \frac{e^{-t}}{1-e^{-t}} T^k \left(\frac{e^{-t}-1}{t} \right) dt \,.
\end{aligned} \tag{28}$$

We apply now (28) with $k = 1$. This gives

$$F_1(q) = \gamma\zeta(q) - \sum_{n\geq 1} \frac{H_{n-1}}{n^q} + \frac{1}{\Gamma(q)} \int_0^{+\infty} t^{q-1} \frac{e^{-t}}{1-e^{-t}} \mathrm{E}_1(t)\, dt\,,$$

with $\mathrm{E}_1(t) = -\mathrm{Ei}(-t) = \displaystyle\int_t^{+\infty} \frac{e^{-u}}{u}\, du$. Thus

$$F_1(q) = \gamma\zeta(q) - \sum_{n\geq 1} \frac{H_n}{n^q} + \zeta(q+1) + I(q)\,,$$

where

$$I(q) = \frac{1}{\Gamma(q)} \int_0^{+\infty} t^{q-1} \frac{e^{-t}}{1-e^{-t}} \mathrm{E}_1(t)\, dt = \frac{1}{\Gamma(q)} \sum_{n=1}^{\infty} \int_0^{+\infty} e^{-nt} t^{q-1} \mathrm{E}_1(t)\, dt\,.$$

Since

$$\mathrm{E}_1(t) = -\gamma - \log t + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{t^n}{n!},$$

and $-\gamma - \log t = \widehat{\frac{\log x}{x}}$ (cf. [13]), then $\mathrm{E}_1 = \widehat{\frac{\log(x+1)}{x}}$. Thus

$$\int_0^{+\infty} e^{-nt} t^{q-1} \mathrm{E}_1(t)\, dt = (-1)^{q-1} \left(\frac{\log(x+1)}{x} \right)^{(q-1)} (n)\,.$$

Hence, by a calculation of the $(q-1)$th derivative, we get

$$I(q) = \frac{(-1)^{q-1}}{(q-1)!} \sum_{n=1}^{\infty} \left(\frac{\log(x+1)}{x} \right)^{(q-1)} (n) = \sum_{n=1}^{\infty} \frac{\log(n+1)}{n^q} - \sum_{k=1}^{q-1} \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{(n+1)^k n^{q-k}}\,.$$

$\square$

Remark 7. 1) We recall *Euler's formula* (cf. [6])

$$\sum_{n=1}^{\infty} \frac{H_n}{n^q} = \begin{cases} \frac{1}{2}(q+2)\zeta(q+1) - \frac{1}{2}\sum_{k=1}^{q-2} \zeta(k+1)\zeta(q-k) & \text{for } q > 2, \\ 2\zeta(3) & \text{for } q = 2. \end{cases}$$

2) From $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)n} = 1$, and the decomposition

$$\frac{1}{(n+1)^k n^{q-k}} = \frac{1}{(n+1)^{k-1} n^{q-k}} - \frac{1}{(n+1)^k n^{q-k-1}} \quad (0 < k < q)\,,$$

the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)^k n^{q-k}}$ may be expressed as a linear combination of zeta values and integers.

Example 9.

$$\sum_{n=1}^{\infty}\frac{\log(n+1)}{n^2}+\gamma\zeta(2)-\zeta(3)-1=\sum_{n=1}^{\infty}\frac{\lambda_n H_n}{n!n^2}\,,$$
$$\sum_{n=1}^{\infty}\frac{\log(n+1)}{n^3}+\gamma\zeta(3)-\frac{1}{10}\zeta(2)^2-\frac{1}{2}\zeta(2)=\frac{1}{2}\sum_{n=1}^{\infty}\frac{\lambda_n H_n^2}{n!n^2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\lambda_n H_n^{(2)}}{n!n^2}\,,$$
$$\sum_{n=1}^{\infty}\frac{\log(n+1)}{n^4}+\gamma\zeta(4)-2\zeta(5)+\zeta(2)\zeta(3)-\frac{2}{3}\zeta(3)+\frac{1}{3}\zeta(2)-\frac{1}{2}=$$
$$\frac{1}{6}\sum_{n=1}^{\infty}\frac{\lambda_n H_n^3}{n!n^2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\lambda_n H_n H_n^{(2)}}{n!n^2}+\frac{1}{3}\sum_{n=1}^{\infty}\frac{\lambda_n H_n^{(3)}}{n!n^2}\,.$$

5.4 Link with the Ramanujan summation

The function F_k has strong connections with the Ramanujan summation (cf. [3], [4]).

If $a\in\mathcal{E}$, then the series $\sum_{n\geq 1}a(n)$ may be written

$$\sum_{n\geq 1}a(n)=\sum_{n\geq 1}\int_0^{+\infty}e^{-nt}\widehat{a}(t)\,dt\,,$$

and a formal permutation of $\sum_{n\geq 1}$ and $\int_0^{+\infty}$ would lead us to write

$$\sum_{n\geq 1}a(n)=\int_0^{+\infty}\frac{1}{1-e^{-t}}e^{-t}\widehat{a}(t)\,dt.$$

However, this last integral may be divergent at 0. Nevertheless we can renormalize it by removing the singularity at zero. This may be done merely by subtracting the polar part $\frac{1}{t}$ of $\frac{1}{1-e^{-t}}$. From Theorem 1, we know that

$$\int_0^{+\infty}(\frac{1}{1-e^{-t}}-\frac{1}{t})e^{-t}\widehat{a}(t)dt=\sum_{n=1}^{\infty}\frac{\lambda_n}{n!}\int_0^{+\infty}e^{-t}(1-e^{-t})^{n-1}\widehat{a}(t)dt=\sum_{n=1}^{\infty}\frac{\lambda_n}{n!}D(a)\,(n)\,.$$

This justifies the following definition.

Definition 11. Let a be a function in $\mathcal{E}=\mathcal{L}(E)$. The *Ramanujan sum* of the series $\sum_{n\geq 1}a(n)$ is defined by

$$\sum_{n\geq 1}^{\mathcal{R}}a(n)=\int_0^{+\infty}(\frac{1}{1-e^{-t}}-\frac{1}{t})e^{-t}\widehat{a}(t)dt=\sum_{n=1}^{\infty}\frac{\lambda_n}{n!}D(a)\,(n)\,. \tag{29}$$

Lemma 3. Let a and b in $\mathcal{E}$. Then

$$\sum_{n\geq 1}^{\mathcal{R}}(a\bowtie b)(n)=\sum_{n=1}^{\infty}\frac{\lambda_n}{n!}D(a)(n)\,D(b)(n)\,. \tag{30}$$

Proof. This results directly from (12) and (29). $\square$

Theorem 11. *for all $s \in \mathbb{C}$ with $\Re(s) \geq 1$, one has*

$$F_0(s) = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^s} \quad \textit{and} \quad F_k(s) = \sum_{n\geq 1}^{\mathcal{R}} \left(\left(\frac{1}{x}\right)^{\bowtie k} \bowtie \frac{1}{x^s} \right)(n) \quad \textit{for } k \geq 1\,. \tag{31}$$

Proof. By (24) and (30), taking into account the invariance of $\frac{1}{x}$ by D, one may write

$$\begin{aligned}
\sum_{n\geq 1}^{\mathcal{R}} \left(\left(\frac{1}{x}\right)^{\bowtie k} \bowtie \frac{1}{x^s} \right)(n) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} D\left(\left(\frac{1}{x}\right)^{\bowtie k} \right)(n) D\left(\frac{1}{x^s}\right)(n) \\
&= \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \left(\frac{1}{x}\right)^{k}(n) D\left(\frac{1}{x^s}\right)(n) \\
&= \sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^k} D\left(\frac{1}{x^s}\right)(n) = F_k(s)\,.
\end{aligned}$$

$\square$

In particular, by (14), one deduces from (31) the following identity.

Corollary 3. *For each natural number k,*

$$F_k(1) = \vartheta(k+1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{1}{n^{k+1}} = \sum_{n\geq 1}^{\mathcal{R}} \frac{P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{n}\,. \tag{32}$$

Example 10.

$$\begin{aligned}
\vartheta(1) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n} = \gamma\,, \\
\vartheta(2) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^2} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n}\,, \\
\vartheta(3) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^3} = \frac{1}{2} \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n^2}{n} + \frac{1}{2} \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n^{(2)}}{n}\,.
\end{aligned}$$

Remark 8. Comparing (32) with

$$F_0(k+1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})\,,$$

one may observe a kind of reciprocity between $F_k(1)$ and $F_0(k+1)$. This results from the fact that $D = D^{-1}$.

Remark 9. In the case $q = 1$, (27) is meaningless since both the series $\sum_{n\geq 1} \dfrac{\log(n+1)}{n}$ and $\sum_{n\geq 1} \dfrac{H_n}{n}$ diverge. However, since

$$\log(x+1) - (\psi(x+1)+\gamma) = \int_0^{+\infty} (e^{-xu}-1)(\frac{1}{1-e^{-u}} - \frac{1}{u})e^{-u}\, du\,,$$

it follows that

$$\left(\frac{\widehat{\log(x+1)}}{x} - \frac{\widehat{\psi(x+1)+\gamma}}{x}\right)(t) = \int_t^{+\infty} \left(\frac{1}{1-e^{-u}} - \frac{1}{u}\right) e^{-u}\, du\,,$$

and then one may easily deduce from (29) the relation

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{\log(n+1)}{n} - \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n} = -\frac{\gamma^2}{2}\,,$$

which may be rewritten in the following form (cf. Example 10):

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{\log(n+1)}{n} = \vartheta(2) - \frac{1}{2}\vartheta(1)^2\,.$$

5.5 Link with the Arakawa-Kaneko zeta function

For $\Re(s) \geq 1$ and $k \geq 1$, one can define in an algebraic fashion the function ξ_k by

$$\xi_k(s) = \sum_{n=1}^{\infty} D\left(\left(\frac{1}{x}\right)^{\bowtie k} \bowtie \frac{1}{x^s}\right)(n) = \sum_{n=1}^{\infty} \frac{1}{n^k} D\left(\frac{1}{x^s}\right)(n). \tag{33}$$

In particular, for all natural numbers m, one has (cf. [8], Corollary 1)

$$\xi_k(m+1) = \sum_{n=1}^{\infty} \frac{1}{n^k} D\left(\frac{1}{x^{m+1}}\right)(n) = \sum_{n=1}^{\infty} \frac{P_m(H_n, H_n^{(2)}, \ldots, H_n^{(m)})}{n^{k+1}}\,.$$

Since $D\left(\dfrac{1}{x^s}\right) = \mathcal{L}\left(\dfrac{\Lambda^{s-1}}{\Gamma(s)}\right)$, one may also rewrite (33) as

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \mathrm{Li}_k(e^{-u})(\Lambda(u))^{s-1} du\,,$$

and the change of variables $t = \Lambda(u)$ leads to the integral representation

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} \mathrm{Li}_k(1-e^{-t})\, dt\,,$$

which is the analogue of (19) (with Li_k in place of f_k) and also the original definition of the Arakawa-Kaneko zeta function (cf. [1], [8]).

Thus, taking into account the facts that $\xi_k(1) = \zeta(k+1)$ and $\mathrm{Li}_1(1-e^{-t}) = t$, and following the same process as in the proof of Theorem 8, one obtains the following analogue of (20):

$$\xi_{k+1}(s) = \zeta(k+1)\zeta(s)+\sum_{j=1}^{k-1}(-1)^j\zeta(k+1-j)Z_j(s)+(-1)^k\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\frac{e^{-t}}{1-e^{-t}}T^k(t)dt\,. \tag{34}$$

In particular, in the simplest case $k = 1$, since

$$T(t) = \int_t^{+\infty}\frac{e^{-u}}{1-e^{-u}}udu = \sum_{m>0}\int_t^{+\infty}e^{-mu}udu = \sum_{m>0}\frac{e^{-tm}}{m}t + \sum_{m>0}\frac{e^{-tm}}{m^2}\,,$$

(34) again gives the formula

$$\xi_2(s) = \zeta(2)\zeta(s) - s\sum_{n>m>0}\frac{1}{n^{s+1}}\frac{1}{m} - \sum_{n>m>0}\frac{1}{n^s}\frac{1}{m^2}$$

already obtained by Arakawa and Kaneko (cf. [1] Theorem 6 (ii)).

6 Conclusion

Most of the general results given for the modified zeta function F_k, especially Theorem 7, Theorem 8, and Theorem 9, also apply (with minor adaptations) to a wide class of functions including the Arakawa-Kaneko zeta function ξ_k, specifically to the class of functions represented by normalized Mellin transforms of type

$$F_{k,\omega}(s) = \frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\frac{e^{-t}}{1-e^{-t}}f_{k,\omega}(1-e^{-t})\,dt$$

with $\omega = (\omega_n)_{n\geq 1}$ and $f_{k,\omega}(z) = \sum_{n=1}^{\infty}\frac{\omega_n}{n^k}z^n$. In particular, under the assumption that $\frac{|\omega_n|}{n^k} = O(\frac{1}{n})$, we have for positive integers m the nice formula

$$F_{k,\omega}(m+1) = \sum_{n=1}^{\infty}\frac{\omega_n}{n^k}D\left(\frac{1}{x^{m+1}}\right)(n) = \sum_{n=1}^{\infty}\omega_n\frac{P_m(H_n, H_n^{(2)},\ldots,H_n^{(m)})}{n^{k+1}}\,,$$

which extends (25). However, this formula is more theoretical than practical because of the fast increase in the size of polynomials P_m: the number of monomials in P_m is equal to the number $p(m)$ of partitions of m, as shown by the explicit expression of the mth modified Bell polynomial.

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