A new class of identities involving Cauchy numbers, harmonic numbers and zeta values

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Preprint submitted
2010

Abstract

Improving an old idea of Hermite, we associate to each natural number k a modified zeta function of order k. The evaluation of the values of these functions F_k at positive integers reveals a wide class of identities linking Cauchy numbers, harmonic numbers and zeta values.

Mathematical Subject Classification (2000) : 11B83, 11M41, 33B15, 40G99.

Keywords : Cauchy numbers, Bell polynomials, Harmonic numbers, Laplace-Borel transform, Mellin transform, zeta values, Ramanujan summation, Hermite's formula.

1 Introduction

It is well known since the second-half of the 19th century that the Riemann zeta function may be represented by the (normalized) Mellin transform

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} \, dt \quad \text{for } \Re(s) > 1 \,,$$

and from late works of Hermite (cf. [10]) that one has also

$$\zeta(s) - \frac{1}{s-1} = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} \left(\sum_{n=1}^{\infty} \frac{\lambda_n}{n!} (1-e^{-t})^n \right) dt \quad \text{for } \Re(s) \geq 1 \,,$$

where $\lambda_1 = \dfrac{1}{2}$ and $\lambda_{n+1} = \displaystyle\int_0^1 x(1-x)\cdots(n-x)\,dx$ are the (non-alternating) Cauchy numbers[1].

Improving Hermite's idea, one may, more generally, consider Mellin transforms of type

$$F(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} f(1-e^{-t})\,dt$$

with $f(z) = \sum_{n=1}^{\infty} \omega_n \dfrac{z^n}{n^k}$ for suitable sequences $(\omega_n)_{n\geq 1}$ of rational numbers. The simplest interesting case $\omega_n = 1$ corresponds to the *Arakawa-Kaneko zeta function* and has been studied in [7]. In this article, we investigate the case $\omega_n = \dfrac{\lambda_n}{n!}$ *i.e.* we study the function

$$F_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} f_k(1-e^{-t})\,dt \text{ with } f_k(z) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{z^n}{n^k} \;(k = 0, 1, 2, \dots)\,,$$

which is *a priori* defined in the half-plane $\Re(s) \geq 1$ but analytically continues in the whole complex s-plane (Theorem 7). We call this function F_k the *modified zeta function of order k*. For $k = 0$, one must keep in mind that $F_0(s)$ is nothing else than $\zeta(s) - \dfrac{1}{s-1}$.

An evaluation by two different ways of the values $F_k(q)$ at positive integers q leads to a new class of identities linking Cauchy numbers, harmonic numbers and zeta values which naturally extends *Hermite's formula* for ζ (cf. [6]) *i.e.*

$$F_0(q) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} P_{q-1}(H_n, H_n^{(2)}, \dots, H_n^{(q-1)}) = \zeta(q) - \frac{1}{q-1}\,,$$

where the polynomials P_m are the *modified Bell polynomials* defined by the generating function

$$\exp\Big(\sum_{k=1}^{\infty} x_k \frac{z^k}{k}\Big) = \sum_{m=0}^{\infty} P_m(x_1, \cdots, x_m)\, z^m\,,$$

and $H_n^{(m)}$ are the harmonic numbers. In the simplest case $k = 1$, this extension of Hermite's formula translates into the following relation (Theorem 10) :

$$F_1(q) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^2} P_{q-1}(H_n, H_n^{(2)}, \dots, H_n^{(q-1)}) =$$

$$\sum_{n=1}^{\infty} \frac{\log(n+1)}{n^q} + \gamma\zeta(q) + \zeta(q+1) - \sum_{n=1}^{\infty} \frac{H_n}{n^q} - \sum_{k=1}^{q-1} \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{(n+1)^k n^{q-k}}\,.$$

[1] These numbers have been introduced for the first time in 1670 by James Gregory in a letter to John Collins.

For example, for $q = 2$, since $P_1(H_n) = H_n$ and $\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3)$, then the previous relation may be written

$$F_1(2) = \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n! n^2} = \sum_{n=1}^{\infty} \frac{\log(n+1)}{n^2} + \gamma\zeta(2) - \zeta(3) - 1 \,,$$

and this generalizes

$$F_0(2) = \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n! n} = \zeta(2) - 1 \,.$$

The function F_k has also an interesting interpretation in terms of Ramanujan summation (cf. [3]) as underscored by Theorem 11. In particular, one shows the identity

$$F_k(1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{1}{n^{k+1}} = \sum_{n\geq 1}^{\mathcal{R}} \frac{P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{n}$$

where, in the right member, $\sum_{n\geq 1}^{\mathcal{R}}$ denotes the sum (in the sense of Ramanujan) of the divergent series. This raises a kind of "duality" between $F_k(1)$ and $F_0(k+1)$.

2 Preliminaries

2.1 The non-alternating Cauchy numbers

Definition 1. The *Cauchy numbers* (cf. [11]) are the rational numbers $\mathscr{C}_m$ defined for all natural numbers m by the exponential generating function :

$$\sum_{m\geq 0} \mathscr{C}_m \frac{z^m}{m!} = \frac{z}{\log(1+z)} \,.$$

Let $\lambda_{n+1} := (-1)^n \mathscr{C}_{n+1}$, then $\lambda_{n+1} > 0$, and changing z in $-z$, we get the following relation

$$\frac{1}{\log(1-z)} + \frac{1}{z} = \sum_{n\geq 0} \frac{\lambda_{n+1}}{(n+1)!} z^n \,. \tag{1}$$

For $z = 1 - e^{-t}$ and $t > 0$, this relation may be rewritten

$$\frac{1}{1-e^{-t}} - \frac{1}{t} = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} (1-e^{-t})^{n-1} \,. \tag{2}$$

For each integer $n \geq 1$, we will call λ_n the nth *non-alternating* Cauchy number.

Example 1. The first non-alternating Cauchy numbers are

$$\lambda_1 = \frac{1}{2} \,, \lambda_2 = \frac{1}{6} \,, \lambda_3 = \frac{1}{4} \,, \lambda_4 = \frac{19}{30} \,, \lambda_5 = \frac{9}{4} \,.$$

2.2 The modified Bell polynomials and the harmonic numbers

Definition 2. The *modified Bell polynomials* (cf. [9]) are the polynomials P_m defined for all natural numbers m by $P_0 = 1$ and the generating function

$$\exp\left(\sum_{k\geq 1} x_k \frac{z^k}{k}\right) = 1 + \sum_{m\geq 1} P_m(x_1, ..., x_m)\, z^m\,. \tag{3}$$

Proposition 1. For all natural numbers m, and each integer $n \geq 1$,

$$\int_0^{+\infty} e^{-t}(1-e^{-t})^{n-1}\frac{t^m}{m!}dt = \frac{P_m(H_n, \dots, H_n^{(m)})}{n} \tag{4}$$

with

$$H_n^{(m)} := \sum_{j=1}^{n} \frac{1}{j^m} \quad \text{and} \quad H_n := H_n^{(1)}\,.$$

Proof. One starts from the classical Euler's relation :

$$\mathrm{B}(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\,du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

and substitute $u = e^{-t}$, $a = 1-z$ and $b = n+1$, then one obtains

$$\int_0^{+\infty} e^{-t}(1-e^{-t})^n e^{tz}dt = \frac{n!}{(1-z)(2-z)\dots(n+1-z)}\,.$$

Moreover, one has

$$\begin{aligned}
\frac{n!}{(1-z)(2-z)\dots(n+1-z)} &= \frac{n!}{(n+1)!} \times \prod_{j=0}^{n}(1-\frac{z}{j+1})^{-1} \\
&= \frac{1}{(n+1)} \times \exp(-\sum_{j=0}^{n}\log(1-\frac{z}{j+1})) \\
&= \frac{1}{(n+1)} \times \exp(\sum_{j=0}^{n}\sum_{k=1}^{\infty}\frac{z^k}{k(j+1)^k}) \\
&= \frac{1}{(n+1)} \exp(\sum_{k=1}^{\infty} H_{n+1}^{(k)}\frac{z^k}{k}) \\
&= \sum_{m=0}^{\infty} \frac{P_m(H_{n+1}^{(1)}, \dots, H_{n+1}^{(m)})}{n+1}\, z^m \quad \text{(by (3))}.
\end{aligned}$$

Thus (4) results by identification of the term in z^m. □

Example 2. For small values of m, one has

$$P_1(H_n) = H_n\,;\; P_2(H_n, H_n^{(2)}) = \frac{(H_n)^2}{2} + \frac{H_n^{(2)}}{2}\,;$$

$$P_3(H_n, H_n^{(2)}, H_n^{(3)}) = \frac{(H_n)^3}{6} + \frac{H_n H_n^{(2)}}{2} + \frac{H_n^{(3)}}{3}\,.$$

2.3 The Laplace-Borel transformation

We consider the vector space E of complex-valued functions $f \in \mathcal{C}^1(]0, +\infty[)$ such that

for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $|f(t)| \le C_\varepsilon e^{\varepsilon t}$ for all $t \in]0, +\infty[$.

In particular, a function $f \in E$ satisfies the two following properties :

a) for all x with $\Re(x) > 0$, $t \mapsto e^{-xt} f(t)$ is integrable on $]0, +\infty[$,

b) for all β with $0 < \beta < 1$, $t \mapsto |f(t)|\,\frac{1}{t^\beta}$ is integrable on $]0, 1[$.

We recall now some basic properties (cf. [12]) of the Laplace transformation in this frame which is appropriate for our purpose.

Definition 3. Let f be a function in E. The *Laplace transform* $\mathcal{L}(f)$ of f is defined by

$$\mathcal{L}(f)(x) = \int_0^{+\infty} e^{-xt} f(t)\, dt \quad \text{for } \Re(x) > 0\,.$$

Proposition 2 (cf. [12])**.** Let $\mathcal{E} := \mathcal{L}(E)$ be the image of E under $\mathcal{L}$. If a is a function in $\mathcal{E}$, then

a) a is an analytic function of x in the half-plane $\Re(x) > 0$.

b) $a(x) \to 0$ when $\Re(x) \to +\infty$.

c) $\mathcal{L} : E \to \mathcal{E}$ is an isomorphism.

Definition 4. Let $a \in \mathcal{E}$. The *Borel transform* of a is the unique function $\widehat{a} \in E$ such that $a = \mathcal{L}(\widehat{a})$. One has the two reciprocal formulas

$$\widehat{a}(t) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} e^{zt} a(z)\, dz \quad \text{for all } c > 0 \text{ and } t > 0\,,$$

and

$$a(x) = \int_0^{+\infty} e^{-xt} \widehat{a}(t) dt \quad \text{for } \Re(x) > 0\,.$$

Definition 5. Let f and g be two functions in E. The convolution product $f * g$ of f and g is the function defined for all $t > 0$ by

$$(f * g)(t) = \int_0^t f(u) g(t-u)\, du\,.$$

Proposition 3 (cf. [12])**.** If $f \in E$ and $g \in E$, then $f * g \in E$ and

$$\mathcal{L}(f * g) = \mathcal{L}(f)\,\mathcal{L}(g)\,. \tag{5}$$

Hence, if $a \in \mathcal{E}$ and $b \in \mathcal{E}$ then $ab \in \mathcal{E}$ since $ab = \mathcal{L}(\widehat{a} * \widehat{b})$.

Theorem 1. *Let a be a function in $\mathcal{E}$. Then the series*

$$\sum_{n\geq 1} \frac{\lambda_n}{n!} \int_0^{+\infty} e^{-t}(1-e^{-t})^{n-1}\widehat{a}(t)dt$$

converges and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \int_0^{+\infty} e^{-t}(1-e^{-t})^{n-1}\widehat{a}(t)dt = \int_0^{+\infty} (\frac{1}{1-e^{-t}} - \frac{1}{t})e^{-t}\widehat{a}(t)dt\,. \tag{6}$$

Proof. By (2)

$$\int_0^{+\infty} (\frac{1}{1-e^{-t}} - \frac{1}{t})e^{-t}\widehat{a}(t)dt = \int_0^{+\infty} \sum_{n=1}^{\infty} \frac{\lambda_n}{n!}(1-e^{-t})^{n-1}e^{-t}\widehat{a}(t)\,dt\,.$$

In the right member, the order of $\int_0^{+\infty}$ and $\sum_{n=1}^{\infty}$ may be interchanged since

$$\begin{aligned}\int_0^{+\infty} \sum_{n=1}^{\infty} \left|\frac{\lambda_n}{n!}(1-e^{-t})^{n-1}e^{-t}\widehat{a}(t)\right| dt &= \int_0^{+\infty} \sum_{n=1}^{\infty} \frac{\lambda_n}{n!}(1-e^{-t})^{n-1}e^{-t}\,|\widehat{a}(t)|\,dt \\ &= \int_0^{+\infty} (\frac{1}{1-e^{-t}} - \frac{1}{t})e^{-t}\,|\widehat{a}(t)|\,dt\end{aligned}$$

and the convergence of this last integral follows from the assumption that $a \in \mathcal{E}$. □

Example 3. Let $a(x) = \frac{1}{x^s}$ with $\Re(s) \geq 1$. Then $a \in \mathcal{E}$ and $\widehat{a}(t) = \dfrac{t^{s-1}}{\Gamma(s)}$. Hence

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \int_0^{+\infty} e^{-t}(1-e^{-t})^{n-1}\frac{t^{s-1}}{\Gamma(s)}dt &= \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-t}(\frac{1}{1-e^{-t}} - \frac{1}{t})t^{s-1}dt \\ &= \begin{cases} \gamma & \text{if } s = 1 \\ \zeta(s) - \dfrac{1}{s-1} & \text{if } s \neq 1 \end{cases}\end{aligned}$$

where γ refers to the Euler constant. In particular, since

$$\int_0^{+\infty} e^{-t}(1-e^{-t})^{n-1}dt = \frac{1}{n} \quad \text{for each integer } n \geq 1\,,$$

then

$$\gamma = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!}\frac{1}{n}\,.$$

3 The operator D

Proposition 4. If $a \in \mathcal{E}$, then the integral

$$\int_0^{+\infty} e^{-t}(1-e^{-t})^{x-1}\widehat{a}(t)dt$$

converges for all x with $\Re(x) > 0$.

Proof. If $a \in \mathcal{E}$ and $\Re(x) > 0$, we may write for $t \in]0, +\infty[$,

$$\left|e^{-t}(1-e^{-t})^{x-1}\widehat{a}(t)\right| \le e^{-t}e^{(1-\Re(x))(-\log(1-e^{-t}))}\left|\widehat{a}(t)\right| .$$

The convergence when $t \to +\infty$ results from the inequality

$$e^{-t}e^{(1-\Re(x))(-\log(1-e^{-t}))}\left|\widehat{a}(t)\right| \le \frac{e^{-t}}{1-e^{-t}}\left|\widehat{a}(t)\right| \le 2e^{-t}\left|\widehat{a}(t)\right| .$$

The convergence when $t \to 0$ results from the inequality

$$e^{(1-\Re(x))(-\log(1-e^{-t}))} \le \begin{cases} 1 & \text{si } \Re(x) \ge 1 \\ \frac{1}{(1-e^{-t})^{(1-\Re(x))}} & \text{si } 0 < \Re(x) < 1 \end{cases}$$

since the function $t \mapsto e^{-t}\left|\widehat{a}(t)\right| \frac{1}{(1-e^{-t})^{\beta}}$ is integrable at 0 for $0 < \beta < 1$ by definition of E. □

Definition 6. Let a be a function in $\mathcal{E}$. We call $D(a)$ the function defined for all x with $\Re(x) > 0$ by

$$D(a)(x) = \int_0^{+\infty} e^{-t}(1-e^{-t})^{x-1}\widehat{a}(t)dt\,. \tag{7}$$

Remark 1. a) By Theorem 1, the series $\sum_{n\ge1} \frac{\lambda_n}{n!}D(a)(n)$ converges and its sum is given by formula (6).

b) The values of $D(a)$ at positive integers may be computed directly without the recourse to $\widehat{a}$. The development of $(1-e^{-t})^n$ by the binomial theorem gives

$$D(a)(n+1) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}a(k+1) \quad \text{for all integer } n \ge 0. \tag{8}$$

Definition 7. We call Λ the C^1-diffeomorphism of $\mathbb{R}_+$ defined by $\Lambda(u) := -\log(1-e^{-u})$. In particular, it is important to note that Λ is involutive :

$$\Lambda^{-1} = \Lambda\,.$$

Theorem 2. *Let a be a function in $\mathcal{E}$. Then the function $D(a) \in \mathcal{E}$ and, moreover, verifies the relation*

$$\widehat{D(a)} = \widehat{a}(\Lambda) \tag{9}$$

where $\widehat{a}(\Lambda)$ denotes $\widehat{a} \circ \Lambda$.

Proof. The change of variables $t = \Lambda(u)$ in (7) gives

$$D(a)(x) = \int_0^{+\infty} e^{-xu}\widehat{a}(\Lambda(u))\, du \quad \text{for } \Re(x) > 0.$$

Thus, $D(a) = \mathcal{L}(\widehat{a}(\Lambda))$. It remains to prove that $D(a) \in \mathcal{E}$. One has only to check that the function $\widehat{a}(\Lambda)$ is in E. This function being in $\mathcal{C}^1(]0,+\infty[)$, it suffices to show that for all $\varepsilon > 0$, the function $u \mapsto e^{-\varepsilon u} |\widehat{a}(-\log(1-e^{-u}))|$ is bounded on $]0,+\infty[$. This results from the existence of $C_\varepsilon > 0$ such that

$$\left|\widehat{a}(-\log(1-e^{-u}))\right| \leq C_\varepsilon (1-e^{-u})^\varepsilon \text{ for all } u \in \,]0,+\infty[\;.$$

□

Example 4. Let $a(x) = \frac{1}{x^s}$ with $\Re(s) \geq 1$. Then $\widehat{a}(t) = \dfrac{t^{s-1}}{\Gamma(s)}$. Thus, by (9),

$$D(\frac{1}{x^s}) = \mathcal{L}\left(\frac{\Lambda^{s-1}}{\Gamma(s)}\right), \tag{10}$$

and if $s = m+1$ whith m a natural number and $n \geq 1$, then by (4),

$$D(\frac{1}{x^{m+1}})(n) = \frac{P_m(H_n, \ldots, H_n^{(m)})}{n}. \tag{11}$$

Remark 2. Theorem 2 may be summarized in the following diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{D} & \mathcal{E} \\ \big\downarrow \mathcal{L}^{-1} & & \big\uparrow \mathcal{L} \\ E & \xrightarrow{\Lambda^\star} & E \end{array}$$

where $\Lambda^\star(\widehat{a}) := \widehat{a}(\Lambda)$. The algebraic properties of D are sum up in the following theorem.

Theorem 3. *The operator D is an automorphism of $\mathcal{E}$ which verifies $D = D^{-1}$ and lets the function $x \mapsto \dfrac{1}{x}$ invariant.*

Proof. We can write $D = \mathcal{L}\Lambda^\star\mathcal{L}^{-1}$ and $\Lambda^\star$ is an automorphism of E which verifies $\Lambda^\star = (\Lambda^\star)^{-1}$ since $\Lambda = \Lambda^{-1}$. Furthermore

$$D(\frac{1}{x}) = \mathcal{L}(1) = \frac{1}{x}.$$

□

4 The harmonic product

Our aim is to define the harmonic product of two functions a and b in $\mathcal{E}$ as being the unique function f of $\mathcal{E}$ such that

$$D(a)\,(x).D(b)\,(x) = D(f)\,(x)\,.$$

Thus, we have to establish that such a function exists and is unique. In order to do this, we introduce first a Λ-convolution product of two functions in E.

4.1 The Λ-convolution product

Proposition 5. If a and b are in $\mathcal{E}$, then $\widehat{a}(\Lambda) * \widehat{b}(\Lambda) \in E$.

Proof. From the definition of the convolution product, one may write

$$\left(\widehat{a}(\Lambda) * (\widehat{b}(\Lambda)\right)(t) = \int_0^t \widehat{a}(\Lambda(u))\widehat{b}(\Lambda(t-u))du\,.$$

Now, for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ and $D_\varepsilon > 0$ such that

$$\begin{aligned} \left|\widehat{a}(-\log(1-e^{-u}))\right| &\leq C_\varepsilon(1-e^{-u})^\varepsilon \text{ and} \\ \left|\widehat{b}(-\log(1-e^{-(t-u)}))\right| &\leq D_\varepsilon(1-e^{-(t-u)})^\varepsilon \text{ for all } u \in\,]0,+\infty[\,. \end{aligned}$$

It follows that

$$\left|(\widehat{a}(\Lambda) * \widehat{b}(\Lambda))(t)\right| \leq C_\varepsilon D_\varepsilon \int_0^t (1-e^{-u})^\varepsilon(1-e^{-(t-u)})^\varepsilon du\,.$$

One has also

$$\begin{aligned} \int_0^t (1-e^{-u})^\varepsilon(1-e^{-(t-u)})^\varepsilon du &= \left(1-e^{-t}\right)^{1+2\varepsilon}\int_0^1 u^\varepsilon\,(1-u)^\varepsilon\,\frac{1}{(1-(1-e^{-t})u)^{\varepsilon+1}}\,du \\ &\leq \left(1-e^{-t}\right)^{1+2\varepsilon}\int_0^1 \frac{1}{(1-(1-e^{-t})u)^{\varepsilon+1}}\,du \leq \left(1-e^{-t}\right)^{1+2\varepsilon}\frac{e^{t\varepsilon}-1}{(1-e^{-t})\varepsilon} \\ &\leq \left(1-e^{-t}\right)^{2\varepsilon}\frac{e^{t\varepsilon}-1}{\varepsilon} \leq \frac{e^{t\varepsilon}}{\varepsilon}. \end{aligned}$$

Hence, $\left|(\widehat{a}(\Lambda) * \widehat{b}(\Lambda))(t)\right| \leq C_\varepsilon D_\varepsilon \frac{e^{t\varepsilon}}{\varepsilon}$, which proves that this function belongs to E as required. □

Definition 8. Let a and b two functions in $\mathcal{E}$. The Λ-*convolution product* $\widehat{a} \circledast \widehat{b}$ of $\widehat{a}$ and $\widehat{b}$ is defined by

$$\widehat{a} \circledast \widehat{b} = \Lambda^\star(\Lambda^\star(\widehat{a}) * \Lambda^\star(\widehat{b}))$$

(or equivalently since $\Lambda^\star = (\Lambda^\star)^{-1}$)

$$(\widehat{a} \circledast \widehat{b})(\Lambda) = \widehat{a}(\Lambda) * \widehat{b}(\Lambda)\,.$$

Remark 3. The Λ-convolution product inherits of the algebraic properties of the ordinary convolution product *i.e.* bilinearity, commutativity and associativity.

4.2 The harmonic product

Definition 9. Let a and b two functions in $\mathcal{E}$. The *harmonic product* $a \bowtie b$ of a and b is defined by

$$a \bowtie b = \mathcal{L}(\widehat{a} \circledast \widehat{b}) \in \mathcal{E}\,.$$

This construction may be summarized in the following diagram

$$\begin{array}{ccccc}
(a,b) & \longrightarrow & (\widehat{a},\widehat{b}) & \longrightarrow & (\widehat{a}(\Lambda),\widehat{b}(\Lambda)) \\
\downarrow & & \downarrow & & \downarrow \\
a \bowtie b & \longleftarrow & \widehat{a} \circledast \widehat{b} & \longleftarrow & \widehat{a}(\Lambda) * \widehat{b}(\Lambda)
\end{array}$$

Remark 4. The harmonic product inherits of the properties of the Λ-convolution product : it is bilinear, commutative and associative.

Theorem 4. *Let a and b in $\mathcal{E}$. Then,*

$$D(a \bowtie b) = D(a)\,D(b) \tag{12}$$

and

$$D(ab) = D(a) \bowtie D(b)\,. \tag{13}$$

Proof. One knows from Theorem 2 that

$$D = \mathcal{L}\Lambda^\star\mathcal{L}^{-1}\,.$$

Hence

$$D(a \bowtie b) = \mathcal{L}\Lambda^\star\mathcal{L}^{-1}(a \bowtie b) = \mathcal{L}\Lambda^\star(\widehat{a} \circledast \widehat{b}) = \mathcal{L}(\Lambda^\star(\widehat{a}) * \Lambda^\star(\widehat{b}))$$

and it follows from (5) and (9) that

$$\mathcal{L}(\Lambda^\star(\widehat{a}) * \Lambda^\star(\widehat{b})) = \mathcal{L}(\Lambda^\star(\widehat{a}))\mathcal{L}(\Lambda^\star(\widehat{b})) = D(a)\,D(b)$$

which proves (12). Moreover, (12) enables to write

$$D(D(a) \bowtie D(b)) = D^2(a)\,D^2(b) = ab \quad (\text{since } D = D^{-1}),$$

and so

$$D(a\,b) = D^2(D(a) \bowtie D(b)) = D(a) \bowtie D(b)$$

which proves (13). □

Remark 5. The values of $(a \bowtie b)(n)$ may be computed without the recourse to $\widehat{a}$ and $\widehat{b}$. By elementary transformations, it can be shown that

$$(a \bowtie b)(n+1) = \int_0^{+\infty}\int_0^{+\infty} (e^{-t-s})(e^{-t} + e^{-s} - e^{-t}e^{-s})^n \widehat{a}(t)\widehat{b}(s)dtds\,.$$

Hence, if the numbers $C_n^{k,l}$ are defined by

$$(X+Y-XY)^n = \sum_{\substack{0\le k\le n\\ 0\le l\le n}} C_n^{k,l}\, X^k Y^l\,,$$

then, one has the following explicit formula

$$(a \bowtie b)(n+1) = \sum_{\substack{0\le k\le n\\ 0\le l\le n}} C_n^{k,l}\, a(k+1)b(l+1)\,.$$

For small values of n, this enables to compute

$$\begin{aligned}
&(a \bowtie b)(1) = a(1)b(1)\,,\\
&(a \bowtie b)(2) = a(2)b(1) + a(1)b(2) - a(2)b(2)\,,\\
&(a \bowtie b)(3) = a(3)b(1) + a(1)b(3) + 2a(2)b(2) - 2a(3)b(2) - 2a(2)b(3) + a(3)b(3)\,.
\end{aligned}$$

Theorem 5. *Let*

$$\left(\frac{1}{x}\right)^{\bowtie k} := \underbrace{\frac{1}{x} \bowtie \frac{1}{x} \bowtie \cdots \bowtie \frac{1}{x}}_{k} \quad (k=1,2,3,\cdots)$$

where $\frac{1}{x}$ *denotes (improperly) the function* $x \mapsto \frac{1}{x}$. *Then, for all natural numbers* $m \ge 0$,

$$\left(\frac{1}{x}\right)^{\bowtie (m+1)} = D(\frac{1}{x^{m+1}})\,.$$

In particular, for all integers $n \ge 1$,

$$\left(\frac{1}{x}\right)^{\bowtie (m+1)} (n) = \frac{P_m(H_n,\ldots,H_n^{(m)})}{n}. \tag{14}$$

Proof. By (13) we have

$$D(\frac{1}{x^{m+1}}) = D(\underbrace{\frac{1}{x}\cdots\frac{1}{x}}_{m+1}) = \left(D(\frac{1}{x})\right)^{\bowtie (m+1)} = \left(\frac{1}{x}\right)^{\bowtie (m+1)} \text{ since } D(\frac{1}{x}) = \frac{1}{x}.$$

Thus, (14) results from (11). $\square$

4.3 The harmonic property

The following theorem explains the reason why the harmonic product is called "harmonic".

Theorem 6. *Let $a \in \mathcal{E}$. Then*

$$\frac{1}{x} \bowtie a = \frac{A(x)}{x}$$

where A denotes the function defined for $\Re(x) > 0$ by

$$A(x) = \int_0^{+\infty} \frac{e^{-xt} - 1}{e^{-t} - 1} e^{-t} \widehat{a}(t) dt \,.$$

In particular, for each integer $n \geq 1$

$$\left(\frac{1}{x} \bowtie a \right) (n) = \frac{A(n)}{n} = \frac{1}{n} \left(\sum_{k=1}^{n} a(k) \right) . \tag{15}$$

Proof. By the definition of the harmonic product, one has

$$\frac{1}{x} \bowtie a = \mathcal{L}(1 \circledast \widehat{a}) \,.$$

Now :

$$(1 \circledast \widehat{a})(\Lambda(u)) = (1 * \widehat{a}(\Lambda))(u) = \int_0^u \widehat{a}(\Lambda(v)) dv = - \int_{+\infty}^{\Lambda(u)} \widehat{a}(t) \frac{e^{-t}}{1 - e^{-t}} dt$$

(by the change of variables $t = \Lambda(v)$). Hence,

$$(1 \circledast \widehat{a})(u) = \int_u^{+\infty} \widehat{a}(t) \frac{e^{-t}}{1 - e^{-t}} dt$$

Thus, we have

$$\begin{aligned}
\frac{1}{x} \bowtie a &= \int_0^{+\infty} e^{-xu} \left(\int_u^{+\infty} \widehat{a}(t) \frac{e^{-t}}{1 - e^{-t}} dt \right) du \\
&= \int_0^{+\infty} \left(\int_0^t e^{-xu} du \right) \widehat{a}(t) \frac{e^{-t}}{1 - e^{-t}} dt \\
&= \frac{1}{x} \int_0^{+\infty} (1 - e^{-xt}) \widehat{a}(t) \frac{e^{-t}}{1 - e^{-t}} dt \\
&= \frac{A(x)}{x} \,.
\end{aligned}$$

Furthermore, for each integer $n \geq 1$, we have

$$A(n) = \int_0^{+\infty} \frac{e^{-nt} - 1}{e^{-t} - 1} e^{-t} \widehat{a}(t) dt = \sum_{k=1}^{n} a(k) \,.$$

□

Example 5.

$$\frac{1}{x} \bowtie \frac{1}{x} = D(\frac{1}{x^2}) = \mathcal{L}(\Lambda) = \frac{H(x)}{x} \quad \text{with } H(x) := \psi(x+1) + \gamma,$$

ψ denoting the logarithmic derivative of Γ. In particular, for each integer $n \geq 1$

$$\left(\frac{1}{x} \bowtie \frac{1}{x}\right)(n) = \frac{H(n)}{n} = \frac{H_n}{n}\,.$$

Example 6. For $\Re(s) \geq 1$,

$$\frac{1}{x} \bowtie \frac{1}{x^s} = \frac{H^{(s)}(x)}{x}$$

with

$$H^{(s)}(x) := \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{1-e^{-xt}}{1-e^{-t}}\, e^{-t}\, t^{s-1}\, dt\,.$$

For each integer $n \geq 1$,

$$\left(\frac{1}{x} \bowtie \frac{1}{x^s}\right)(n) = \frac{H^{(s)}(n)}{n} = \frac{H_n^{(s)}}{n} = \frac{1}{n}\left(\sum_{m=1}^{n} \frac{1}{m^s}\right).$$

From (15), by induction on k, we deduce the following important corollary

Corollary 1. *For each integer $k \geq 2$,*

$$\left(\left(\frac{1}{x}\right)^{\bowtie k} \bowtie a\right)(n) = \frac{1}{n}\left(\sum_{n \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{a(n_k)}{n_1 \ldots n_{k-1}}\right) \tag{16}$$

Example 7. Applying (16) with $a(x) = \frac{1}{x}$ (and $k = m$), we get

$$\left(\frac{1}{x}\right)^{\bowtie(m+1)}(n) = \frac{1}{n}\left(\sum_{n \geq n_1 \geq \cdots \geq n_m \geq 1} \frac{1}{n_1 \ldots n_m}\right) \tag{17}$$

Hence, it follows from (14) and (17) that

$$P_m(H_n, H_n^{(2)}, \ldots, H_n^{(m)}) = \sum_{n \geq n_1 \geq \cdots \geq n_m \geq 1} \frac{1}{n_1 \ldots n_m}\,, \tag{18}$$

which is a nice reformulation of Dilcher's formula (cf. [2], [8]).

5 The modified zeta function F_k

5.1 Integral representation

Definition 10. For all $s \in \mathbb{C}$ with $\Re(s) \geq 1$ and each natural number k,

$$F_k(s) := \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} f_k(1-e^{-t})\, dt \quad \text{with} \quad f_k(z) := \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{z^n}{n^k}\,. \tag{19}$$

Remark 6. By (2), one has $F_0(s) = \zeta(s) - \dfrac{1}{s-1}$ (cf. Example 3).

The fact that F_k may be represented by a Mellin transform enables to analytically continue this function outside its half-plane of definition by a standard analytic method (cf. [13] section 6.7).

Theorem 7. *The function F_k analytically continues in the whole complex plane as an entire function.*

Proof. The function $z \mapsto \dfrac{1}{\log(1-z)} + \dfrac{1}{z}$ being analytic in the disc $D(0,1)$ with a singularity at 1, we deduce from (1) that the radius of convergence of the series $\sum_{n=1}^{\infty} \dfrac{\lambda_n z^n}{n!}$ is equal to 1. Thus 1 is also the radius of convergence of the serie $\sum_{n=1}^{\infty} \dfrac{\lambda_n z^n}{n! n^k}$ which defines an analytic function f_k in the disc $D(0,1)$. Hence, the function

$$g_k : t \mapsto f_k(1-e^{-t})$$

is analytic for all $t \in \mathbb{C}$ such that $1-e^{-t} \in D(0,1)$. Since $1-e^0 = 0$, it follows that g_k is analytic in a neighbourhood of 0. Since $g_k(0) = 0$, the function $t \mapsto g_k(t)\dfrac{e^{-t}}{1-e^{-t}}$ is itself analytic in a neighbourhood of 0. It follows that its Mellin transform analytically continues in the complex plane with simple poles at negative integers which are all cancelled by the poles of Γ. ∎

Theorem 8. *For all s with $\Re(s) > 1$ and each integer $k \geq 1$,*

$$F_k(s) = \vartheta(k)\zeta(s) + \sum_{j=1}^{k} (-1)^j \vartheta(k-j) Z_j(s) + (-1)^k \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} T^k \left(\frac{e^{-t}-1}{t} \right) dt \tag{20}$$

with

$$\vartheta(k) := \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{1}{n^k}\,, \tag{21}$$

$$Z_j(s) := \sum_{n > n_1 > n_2 > \cdots > n_j > 0} \frac{1}{n^s n_1 n_2 \ldots n_j}\,, \tag{22}$$

$$Tf\,(t) := \int_t^{+\infty} \frac{e^{-u}}{1-e^{-u}} f(u) du\,. \tag{23}$$

Proof. Formula (20) results from the integral representation (19) and the two following lemmas.

Lemma 1. For all $t > 0$,

$$f_k(1-e^{-t}) = \sum_{j=0}^{k} (-1)^j \vartheta(k-j) \frac{\Lambda^j(t)}{j!} + (-1)^k T^k\left(\frac{e^{-t}-1}{t}\right)$$

where ϑ is defined by (21) and T is the operator defined by (23).

Proof. Let $g_k(t) := f_k(1-e^{-t})$. The function g_k verifies the recursive relation

$$g_k'(t) = e^{-t} f_k'(1-e^{-t}) = \frac{e^{-t}}{1-e^{-t}} f_{k-1}(1-e^{-t}) = \frac{e^{-t}}{1-e^{-t}} g_{k-1}(t)$$

Thus

$$g_k(t) = \int_0^t \frac{e^{-u}}{1-e^{-u}} g_{k-1}(u)du = g_k(+\infty) - \int_t^{+\infty} \frac{e^{-u}}{1-e^{-u}} g_{k-1}(u)du$$

with

$$g_k(+\infty) = f_k(1) = \vartheta(k).$$

Thus, one has

$$g_k(t) = \vartheta(k) - \int_t^{+\infty} \frac{e^{-u}}{1-e^{-u}} g_{k-1}(u)du = \vartheta(k) - T(g_{k-1})\,.$$

A repeated iteration k times of this relation gives

$$g_k(t) = \sum_{j=0}^{k-1} \vartheta(k-j)(-1)^j T^j\,(1) + (-1)^k T^k(g_0)\,.$$

Now, by (2),

$$g_0(t) = \sum_{n=1}^{\infty} \frac{\lambda_n (1-e^{-t})^n}{n!} = \frac{e^{-t}-1}{t} + 1\,,$$

and thus

$$T^k(g_0) = T^k\left(\frac{e^{-t}-1}{t}\right) + T^k(1)\,.$$

Hence

$$g_k(t) = \sum_{j=0}^{k-1} \vartheta(k-j)(-1)^j T^j\,(1) + (-1)^k T^k(1) + (-1)^k T^k\left(\frac{e^{-t}-1}{t}\right).$$

Since $\vartheta(0) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} = 1$ (by (1) and a tauberian theorem), one deduces that

$$g_k(t) = \sum_{j=0}^{k} \vartheta(k-j)(-1)^j T^j\,(1) + (-1)^k T^k\left(\frac{e^{-t}-1}{t}\right)$$

and, now, it remains to prove that

$$\frac{\Lambda^j(t)}{j!} = T^j(1)$$

which follows from the recursive relation

$$\frac{\Lambda^j(t)}{j!} = -\int_{+\infty}^{t} \frac{e^{-u}}{1-e^{-u}} \frac{\Lambda^{j-1}(u)}{(j-1)!} du = T\left(\frac{\Lambda^{j-1}}{(j-1)!}\right).$$

□

Lemma 2. Let $Z_j(s)$ defined by (22). Then, for all $s \in \mathbb{C}$ with $\Re(s) > 1$,

$$Z_j(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} \frac{\Lambda^j(t)}{j!} dt\,.$$

Proof. From the recursive relation

$$\partial \frac{\Lambda^j(t)}{j!} = \frac{\Lambda^{j-1}(t)}{(j-1)!} \partial\Lambda(t) = -\frac{e^{-t}}{1-e^{-t}} \frac{\Lambda^{j-1}(t)}{(j-1)!} = -\sum_{m>0} e^{-mt} \frac{\Lambda^{j-1}(t)}{(j-1)!}\,,$$

and $\Lambda(t) = \sum_{n>0} \frac{e^{-nt}}{n}$, one may check by induction on j that

$$\frac{\Lambda^j(t)}{j!} = \sum_{n_1>n_2>\ldots>n_j>0} \frac{e^{-n_1 t}}{n_1} \frac{1}{n_2} \cdots \frac{1}{n_j}\,.$$

Furthermore, one has

$$\frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} e^{-Nt} \frac{e^{-t}}{1-e^{-t}} dt = \sum_{n>N} \frac{1}{n^s} \quad (\text{for } \Re(s) > 1)\,.$$

Hence

$$\frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} \frac{\Lambda^j(t)}{j!} dt = \sum_{n>n_1>n_2>\cdots>n_j>0} \frac{1}{n^s} \frac{1}{n_1} \frac{1}{n_2} \cdots \frac{1}{n_j} = Z_j(s)\,.$$

□

□

5.2 Values of F_k at integers

Theorem 9. *For all s in $\mathbb{C}$ with $\Re(s) \geq 1$ and each natural number k, then*

$$F_k(s) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^k} D\left(\frac{1}{x^s}\right)(n)\,. \tag{24}$$

In particular, for all natural numbers m,

$$F_k(m+1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^{k+1}} P_m(H_n, H_n^{(2)}, \ldots, H_n^{(m)})\,. \tag{25}$$

Proof. The change of variables $t = \Lambda(u)$ in (19) enables to write

$$F_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} f_k(e^{-u})(\Lambda(u))^{s-1} du\,.$$

Since $D(\frac{1}{x^s}) = \mathcal{L}\left(\frac{\Lambda^{s-1}}{\Gamma(s)}\right)$, we deduce (24) from this last expression of $F_k(s)$. Moreover, by (11), one has $D(\frac{1}{x^{m+1}})(n) = \frac{P_m(H_n, \ldots, H_n^{(m)})}{n}$ which proves (25). □

Corollary 2. *Let $\vartheta(s)$ be the Dirichlet series defined for $\Re(s) > 0$ by*

$$\vartheta(s) := \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \frac{1}{n^s}\,.$$

Then for each natural number k,

$$\vartheta(k+1) = F_k(1)\,. \tag{26}$$

Example 8.

$$\begin{aligned}
F_0(1) =& \sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} = \gamma = \vartheta(1)\,,\\
F_0(2) =& \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n! n} = \zeta(2) - 1\,,\\
F_0(3) =& \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^2}{n! n} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n! n} = \zeta(3) - \frac{1}{2}\,,\\
F_1(1) =& \sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^2} = \vartheta(2)\,,\\
F_1(2) =& \sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n! n^2}\,,\\
F_1(3) =& \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^2}{n! n^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n! n^2}\,.
\end{aligned}$$

5.3 Identities linking Cauchy numbers, harmonic numbers and zeta values

Theorem 10. *For all integers $q \geq 2$,*

$$F_1(q) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^2} P_{q-1}(H_n, H_n^{(2)}, \ldots, H_n^{(q-1)}) =$$
$$\sum_{n=1}^{\infty} \frac{\log(n+1)}{n^q} + \gamma\zeta(q) + \zeta(q+1) - \sum_{n=1}^{\infty} \frac{H_n}{n^q} - \sum_{k=1}^{q-1} \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{(n+1)^k n^{q-k}} . \quad (27)$$

Proof. By (20) and (25), one may write

$$F_k(q) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^{k+1}} P_{q-1}(H_n, H_n^{(2)}, \ldots, H_n^{(q-1)}) =$$
$$\vartheta(k)\zeta(q) + \sum_{j=1}^{k} (-1)^j \vartheta(k-j) Z_j(q) + (-1)^k \frac{1}{\Gamma(q)} \int_0^{+\infty} t^{q-1} \frac{e^{-t}}{1-e^{-t}} T^k \left(\frac{e^{-t}-1}{t} \right) dt \, . \quad (28)$$

We apply now (28) with $k = 1$. This gives

$$F_1(q) = \gamma\zeta(q) - \sum_{n \geq 1} \frac{H_{n-1}}{n^q} + \frac{1}{\Gamma(q)} \int_0^{+\infty} t^{q-1} \frac{e^{-t}}{1-e^{-t}} \mathrm{E}_1(t) dt$$

with $\mathrm{E}_1(t) := -\mathrm{Ei}(-t) = \int_t^{+\infty} \frac{e^{-u}}{u} du$. Thus,

$$F_1(q) = \gamma\zeta(q) - \sum_{n \geq 1} \frac{H_n}{n^q} + \zeta(q+1) + I(q)$$

where

$$I(q) = \frac{1}{\Gamma(q)} \int_0^{+\infty} t^{q-1} \frac{e^{-t}}{1-e^{-t}} \mathrm{E}_1(t) dt = \frac{1}{\Gamma(q)} \sum_{n=1}^{\infty} \int_0^{+\infty} e^{-nt} t^{q-1} \mathrm{E}_1(t) dt \, .$$

Since

$$\mathrm{E}_1(t) = -\gamma - \log t + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{t^n}{n!},$$

and $-\gamma - \log t = \widehat{\frac{\log x}{x}}$ (cf. [12]), then $\mathrm{E}_1 = \widehat{\frac{\log(x+1)}{x}}$. Thus

$$\int_0^{+\infty} e^{-nt} t^{q-1} \mathrm{E}_1(t) dt = (-1)^{q-1} \left(\frac{\log(x+1)}{x} \right)^{(q-1)} (n) \, .$$

Hence, by a calculation of the $(q-1)$th derivative, we get

$$I(q) = \frac{(-1)^{q-1}}{(q-1)!}\sum_{n=1}^{\infty}\left(\frac{\log(x+1)}{x}\right)^{(q-1)}(n) = \sum_{n=1}^{\infty}\frac{\log(n+1)}{n^q} - \sum_{k=1}^{q-1}\frac{1}{k}\sum_{n=1}^{\infty}\frac{1}{(n+1)^k n^{q-k}}\,.$$

□

Remark 7. 1) We recall *Euler's formula* (cf. [5])

$$\sum_{n=1}^{\infty}\frac{H_n}{n^q} = \begin{cases} \frac{1}{2}(q+2)\zeta(q+1) - \frac{1}{2}\sum_{k=1}^{q-2}\zeta(k+1)\zeta(q-k) & \text{for } q>2 \\ 2\zeta(3) & \text{for } q=2 \end{cases}$$

2) From $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+1)n} = 1$ and the decomposition

$$\frac{1}{(n+1)^k n^{q-k}} = \frac{1}{(n+1)^{k-1}n^{q-k}} - \frac{1}{(n+1)^k n^{q-k-1}} \quad (0<k<q)\,,$$

the sum of the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+1)^k n^{q-k}}$ may be expressed as a linear combination of zeta values and integers.

Example 9.

$$\sum_{n=1}^{\infty}\frac{\log(n+1)}{n^2} + \gamma\zeta(2) - \zeta(3) - 1 = \sum_{n=1}^{\infty}\frac{\lambda_n H_n}{n!n^2}\,,$$

$$\sum_{n=1}^{\infty}\frac{\log(n+1)}{n^3} + \gamma\zeta(3) - \frac{1}{10}\zeta(2)^2 - \frac{1}{2}\zeta(2) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\lambda_n H_n^2}{n!n^2} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{\lambda_n H_n^{(2)}}{n!n^2}\,,$$

$$\sum_{n=1}^{\infty}\frac{\log(n+1)}{n^4} + \gamma\zeta(4) - 2\zeta(5) + \zeta(2)\zeta(3) - \frac{2}{3}\zeta(3) + \frac{1}{3}\zeta(2) - \frac{1}{2} =$$

$$\frac{1}{6}\sum_{n=1}^{\infty}\frac{\lambda_n H_n^3}{n!n^2} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{\lambda_n H_n H_n^{(2)}}{n!n^2} + \frac{1}{3}\sum_{n=1}^{\infty}\frac{\lambda_n H_n^{(3)}}{n!n^2}\,.$$

5.4 Link with the Ramanujan summation

The function F_k has strong connections with Ramanujan summation (cf. [3], [4]).

Definition 11. Let a be a function in $\mathcal{E} = \mathcal{L}(E)$. The *Ramanujan sum* of the series $\sum_{n\geq 1} a(n)$ is defined by

$$\sum_{n\geq 1}^{\mathcal{R}} a(n) := \sum_{n=1}^{\infty}\frac{\lambda_n}{n!}D(a)(n)\,. \tag{29}$$

Lemma 3. Let a and b in $\mathcal{E}$. Then

$$\sum_{n\geq 1}^{\mathcal{R}}(a \bowtie b)(n) = \sum_{n=1}^{\infty}\frac{\lambda_n}{n!}D(a)(n)\,D(b)(n)\,. \tag{30}$$

Proof. This results directly from (12) and (29). □

Theorem 11. *for all $s \in \mathbb{C}$ with $\Re(s) \geq 1$, one has*

$$F_0(s) = \sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n^s} \quad \textit{and} \quad F_k(s) = \sum_{n\geq 1}^{\mathcal{R}}\left(\left(\frac{1}{x}\right)^{\bowtie k} \bowtie \frac{1}{x^s}\right)(n) \quad \textit{for } k \geq 1\,. \tag{31}$$

Proof. By (24) and (30), taking into account the invariance of $\frac{1}{x}$ by D, one may write

$$\begin{aligned}\sum_{n\geq 1}^{\mathcal{R}}\left(\left(\frac{1}{x}\right)^{\bowtie k} \bowtie \frac{1}{x^s}\right)(n) &= \sum_{n=1}^{\infty}\frac{\lambda_n}{n!}D\left(\left(\frac{1}{x}\right)^{\bowtie k}\right)(n)D\left(\frac{1}{x^s}\right)(n)\\ &= \sum_{n=1}^{\infty}\frac{\lambda_n}{n!}\left(\frac{1}{x}\right)^{k}(n)D\left(\frac{1}{x^s}\right)(n)\\ &= \sum_{n=1}^{\infty}\frac{\lambda_n}{n!n^k}D\left(\frac{1}{x^s}\right)(n) = F_k(s)\,.\end{aligned}$$

□

In particular, by (14), one deduces from (31) the following identity :

Corollary 3. *For each natural number k,*

$$F_k(1) = \vartheta(k+1) = \sum_{n=1}^{\infty}\frac{\lambda_n}{n!}\frac{1}{n^{k+1}} = \sum_{n\geq 1}^{\mathcal{R}}\frac{P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{n}\,. \tag{32}$$

Example 10.

$$\begin{aligned}\vartheta(1) &= \sum_{n=1}^{\infty}\frac{\lambda_n}{n!n} = \sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n} = \gamma\,,\\ \vartheta(2) &= \sum_{n=1}^{\infty}\frac{\lambda_n}{n!n^2} = \sum_{n\geq 1}^{\mathcal{R}}\frac{H_n}{n}\,,\\ \vartheta(3) &= \sum_{n=1}^{\infty}\frac{\lambda_n}{n!n^3} = \frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}}\frac{H_n^2}{n} + \frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}}\frac{H_n^{(2)}}{n}\,.\end{aligned}$$

Remark 8. Comparing (32) with

$$F_0(k+1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} P_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)}) \,,$$

one may observe a kind of duality between $F_k(1)$ and $F_0(k+1)$. This results from the fact that $D = D^{-1}$.

Remark 9. In the case $q = 1$, (27) is meaningless since both the series $\sum_{n\geq 1} \frac{\log(n+1)}{n}$ and $\sum_{n\geq 1} \frac{H_n}{n}$ diverge. However, one may easily show the relation

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{\log(n+1)}{n} + \frac{\gamma^2}{2}$$

which may be rewritten under the following form

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{\log(n+1)}{n} = \vartheta(2) - \frac{1}{2}\vartheta(1)^2 \,.$$

5.5 Link with the Arakawa-Kaneko zeta function

For $\Re(s) \geq 1$ and $k \geq 1$, one can define in an algebraic fashion a function ξ_k by

$$\xi_k(s) := \sum_{n=1}^{\infty} D\left(\left(\frac{1}{x}\right)^{\bowtie k} \bowtie \frac{1}{x^s}\right)(n) = \sum_{n=1}^{\infty} \frac{1}{n^k} D\left(\frac{1}{x^s}\right)(n). \tag{33}$$

In particular, one has for positive integers m

$$\xi_k(m+1) = \sum_{n=1}^{\infty} \frac{1}{n^k} D\left(\frac{1}{x^{m+1}}\right)(n) = \sum_{n=1}^{\infty} \frac{P_m(H_n, H_n^{(2)}, \ldots, H_n^{(m)})}{n^{k+1}} \,.$$

Since $D\left(\frac{1}{x^s}\right) = \mathcal{L}\left(\frac{\Lambda^{s-1}}{\Gamma(s)}\right)$, one may also rewrite (33) as

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \mathrm{Li}_k(e^{-u})(\Lambda(u))^{s-1} du \,,$$

and the change of variables $t = \Lambda(u)$ leads to the integral representation

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} \mathrm{Li}_k(1-e^{-t})\, dt$$

which is the analogue of (19) (with Li_k in place of f_k) and also the original definition of the Arakawa-Kaneko zeta function (cf. [1], [7]).

Thus, taking in account the facts that $\xi_k(1) = \zeta(k+1)$ and $\mathrm{Li}_1(1-e^{-t}) = t$, and following the same process as in the proof of Theorem 8, one obtains the following analogue of (20) :

$$\xi_{k+1}(s) = \sum_{j=0}^{k-1} (-1)^j \zeta(k+1-j) Z_j(s) + (-1)^k \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} T^k(t) dt \qquad (34)$$

In particular, in the simplest case $k = 1$, since

$$T(t) = \int_t^{+\infty} \frac{e^{-u}}{1-e^{-u}} u du = \sum_{m>0} \int_t^{+\infty} e^{-mu} u du = \sum_{m>0} \frac{e^{-tm}}{m} t + \sum_{m>0} \frac{e^{-tm}}{m^2} ,$$

(34) translates into the formula

$$\xi_2(s) = \zeta(2)\zeta(s) - s \sum_{n>m>0} \frac{1}{n^{s+1}} \frac{1}{m} - \sum_{n>m>0} \frac{1}{n^s} \frac{1}{m^2}$$

already obtained by Arakawa and Kaneko (cf. [1] Theorem 6 (ii)).

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