The Method of Strained Coordinates for Vibrations with Weak Unilateral Springs

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Abstract

We study some spring mass models for a structure having a unilateral spring of small rigidity ε. We obtain and justify an asymptotic expansion with the method of strained coordinates with new tools to handle such defects, including a non negligible cumulative effect over a long time: $T_\varepsilon \sim \varepsilon^{-1}$ as usual; or, for a new critical case, we can only expect: $T_\varepsilon \sim \varepsilon^{-1/2}$. We check numerically these results and present a purely numerical algorithm to compute "Non linear Normal Modes" (NNM); this algorithm provides results close to the asymptotic expansions but enables to compute NNM even when ϵ becomes larger.

Keywords: nonlinear vibrations, method of strained coordinates, piecewise linear, unilateral spring, approximate nonlinear normal mode.

Mathematics Subject Classification. Primary: 34E15;
Secondary: 26A16, 26A45, 41A80.

1 Introduction

For spring mass models, the presence of a small piecewise linear rigidity can model a small defect which implies unilateral reactions on the structure. So, the nonlinear and piecewise linear function $u_+ = \max(0, u)$ plays a key role in this paper. For nondestructive testing we study a singular nonlinear effect for large time by asymptotic expansion of the vibrations. New features and comparisons with classical cases of smooth perturbations are given, for instance, with the classical Duffing equation: $\ddot{u} + u + \varepsilon u^3 = 0$ and the non classical case: $\ddot{u} + u + \varepsilon u_+ = 0$. Indeed, piecewise linearity is singular: nonlinear and Lipschitz but not differentiable. We give some new results to validate such asymptotic expansions. Furthermore, these tools are also valid for a more general non linearity. A nonlinear crack approach for elastic waves can be found in [11]. Another approach in the framework of non-smooth analysis can be found in [2, 4, 19].
For short time, a linearization procedure is enough to compute a good approximation. But for large time, nonlinear cumulative effects drastically alter the nature of the solution. We will consider the classical method of strained coordinates to compute asymptotic expansions. The idea goes further back to Stokes, who in 1847 calculated periodic solutions for a weakly nonlinear wave propagation problem, see [15, 16, 17, 18] for more details and references therein. Subsequent authors have generally referred to this as the Poincaré method or the

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Lindstedt method. It is a simple and efficient method which gives us approximate nonlinear normal modes with 1 or more degrees of freedom.
In section 2 we present the method on an explicit case with an internal Lipschitz force. We focus on an equation with one degree of freedom with expansions valid for time of order ε^{-1} or, more surprisingly, $\varepsilon^{-1/2}$ for a degenerate contact.
Section 3 contains a tool to expand $(u+\varepsilon v)_+$ and some accurate estimate for the remainder. This is a new key point to validate the method of strained coordinates with unilateral contact.
In Section 4, we extend previous results to systems with N degrees of freedom, first, with the same accuracy for approximate nonlinear normal modes, then, with less accuracy with all modes. We check numerically these results and present a purely numerical algorithm to compute "Non linear normal Modes" (NNM) in the sense of Rosenberg [22]; see [1] for two methods for the computation of NNM; see [9] for a computation of non linear normal mode with unilateral contact and [14] for a synthesis on non linear normal modes; this algorithm provides results close to the asymptotic expansions but enables to compute NNM even when ϵ becomes larger.
In Section 5, we briefly explain why we only perform expansions with even periodic functions to compute the nonlinear frequency shift.
Section 6 is an appendix containing some technical proofs and results.

2 One degree of freedom

2.1 Explicit angular frequency

We consider a one degree of freedom spring-mass system (see figure 1): one spring is classical linear and attached to the mass and to a rigid wall, the second is still linear attached to a rigid wall but has a unilateral contact with the mass; this is to be considered as a damaged spring. The force acting on the mass is $k_1 u + k_2 u_+$ where u is the displacement of the mass, k_1, the rigidity of the undamaged spring and k_2, the rigidity of the damaged unilateral spring. We notice that the term

$$u_+ = \max(0, u).$$

is Lipschitz but not differentiable. Assuming that $k_2 = \varepsilon k_1$ and dividing by the value of the mass we can consider the equation:

$$\ddot{u} + \omega_0^2 u + \varepsilon u_+ = 0, \tag{1}$$

where ω_0 a positive constant. This case has a constant energy E, where

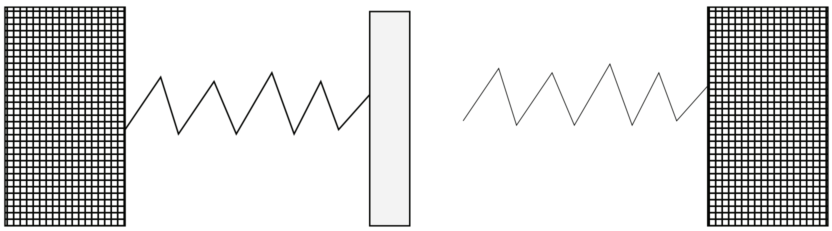

Figure 1: Two springs, on the right it has only a unilateral contact.

$$2E = \dot{u}^2 + \omega_0^2 u^2 + \varepsilon (u_+)^2.$$

Therefore, the level sets of $E(u, \dot{u})$ will be made of two half ellipses. Indeed, for $u < 0$ the level set is an half ellipse, and for $u > 0$ is another half ellipse. Any solution $u(t)$ is confined

to a closed level curve of $E(u,\dot u)$ and is necessarily a periodic function of t.
More precisely, a non trivial solution ($E>0$) is on the half ellipse: $\dot u^2+\omega_0^2 u=2E$, in the phase plane during the time $T_C=\pi/\omega_0$, and on the half ellipse $\dot u^2+(\omega_0^2+\varepsilon)u=2E$ during the time $T_E=\pi/\sqrt{\omega_0^2+\varepsilon}$. The period $P(\varepsilon)$ is then
$P(\varepsilon)=(1+\left(1+\varepsilon/\omega_0^2\right)^{-1/2})\pi/\omega_0$, and the exact angular frequency is:

$$\begin{aligned}\omega(\varepsilon) &= 2\omega_0(1+\left(1+\varepsilon/\omega_0^2\right)^{-1/2})^{-1} \\ &= \omega_0+\frac{\varepsilon}{(4\omega_0)}-\frac{\varepsilon^2}{(8\omega_0^3)}+\mathcal{O}(\varepsilon^3).\end{aligned} \tag{2}$$

Let us compare with the angular frequency for Duffing equation where the nonlinear term is u^3:

$$\ddot u+\omega_0^2 u+\varepsilon u^3=0, \tag{3}$$

which depends on the amplitude a_0 of the solution (see for example [15, 16, 17, 18]):

$$\omega_D(\varepsilon) = \omega_0+\frac{3}{8\omega_0^2}a_0^2\varepsilon-\frac{15}{256\omega_0^4}a_0^4\varepsilon^2+\mathcal{O}(\varepsilon^3).$$

2.2 The method of strained coordinates

Now, we compute, with the method of strained coordinates, ω_ε, *an approximation* of the exact angular frequency $\omega(\varepsilon)$ which is smooth with respect to ε by exact formula (2):

$$\omega(\varepsilon) = \omega_\varepsilon + \mathrm{O}(\varepsilon^3).$$

We expose this case completely to use the same method of strained coordinates later when we will not have such an explicit formula.
Let us define the new time and rewrite equation (1) with the new time

$$s=\omega_\varepsilon t, \qquad u_\varepsilon(t)=v_\varepsilon(s), \tag{4}$$
$$\omega_\varepsilon^2 v_\varepsilon''(s)+\omega_0^2 v_\varepsilon(s)+\varepsilon(v_\varepsilon(s))_+=0, \tag{5}$$

To simplify the exposition and the computations, we take following initial conditions for u_ε

$$u_\varepsilon(0)=a_0>0, \qquad \dot u_\varepsilon(0) = 0, \tag{6}$$

i.e. $v_\varepsilon(0)=a_0$ and $v_\varepsilon'(0)=0$. Similar computations are valid for negative a_0, see Proposition 2.1 below. With more general data, i.e. when $\dot u_\varepsilon(0)\neq 0$, computations are more complicate and give the same angular frequency ω_ε, see section 5.
In new time s, we use the following **ansatz**

$$v_\varepsilon(s) = v_0(s)+\varepsilon v_1(s)+\varepsilon^2 r_\varepsilon(s). \tag{7}$$

and the following notations:

$$\omega_\varepsilon = \omega_0+\varepsilon\omega_1+\varepsilon^2\omega_2, \quad \omega_\varepsilon^2=\alpha_0+\varepsilon\alpha_1+\varepsilon^2\alpha_2+\mathcal{O}(\varepsilon^3) \tag{8}$$
$$\alpha_0 = \omega_0^2,\ \alpha_1=2\omega_0\omega_1, \quad \alpha_2=\omega_1^2+2\omega_0\omega_2, \tag{9}$$

where ω_1,ω_2 or α_1,α_2 are unknown.
We will also use the natural expansion, $(u+\varepsilon v)_+ = u_+ + \varepsilon H(u)v+\cdots$, where H is the Heaviside function, equal to 1 if $u>0$ and else 0. This expansion is validated in Lemma 3.1 below.

Now, replacing this ansatz in (5) we obtain differential equations and initial data for v_0, v_1, r_ε; set

$$L(v) = -\alpha_0(v'' + v) \tag{10}$$

$$\begin{aligned}
L(v_0) &= \quad 0, & v_0(0) = a_0, v_0'(0) = 0, \quad &(11)\\
L(v_1) &= \quad (v_0)_+ + \alpha_1 v_0'', & v_1(0) = 0, \ \ v_1'(0) = 0, \quad &(12)\\
L(r_\varepsilon) &= \quad H(v_0)v_1 + \alpha_2 v_0'' + \alpha_1 v_1'' + R_\varepsilon(s), & r_\varepsilon(0) = 0, \ \ r_\varepsilon'(0) = 0. \quad &(13)
\end{aligned}$$

We now compute, α_1, v_1 and then α_2. We have $v_0(s) = a_0\cos(s)$. A key point in the method of strained coordinates is to keep bounded v_1 and r_ε for large time by a choice of α_1 for u_1 and α_2 for r_ε. For this purpose, we avoid resonant or **secular** term in the right-hand-side of equations (12), (13). Let us first focus on α_1. Notice that, $v_0(s) = a_0\cos(s)$ and $a_0 > 0$, so

$$(v_0)_+ \quad = \quad a_0\left(\frac{\cos s}{2} + \frac{|\cos s|}{2}\right).$$

$|\cos(s)|$ has no term with frequencies ± 1, since there are only even frequencies. Thus $((v_0)_+ - \alpha_1 v_0) = a_0\cos(s)(1/2 - \alpha_1) + a_0|\cos(s)|/2$ has no secular term if and only if $\alpha_1 = 1/2$, so $\omega_1 = 1/(4\omega_0)$. Now, v_1 satisfies:

$$-\omega_0^2(v_1'' + v_1) = a_0|\cos s|/2, \qquad v_1(0) = 0, \quad v_1'(0) = 0.$$

To remove secular term in the equation (13) we have to obtain the Fourier expansion for $H(v_0)$ and v_1. Some computations yield:

$$|\cos(s)| \quad = \quad \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{+\infty}\frac{(-1)^k}{4k^2-1}\cos(2ks), \tag{14}$$

$$v_1(s) \quad = \quad \frac{-a_0}{\pi\omega_0^2}\left(1 - 2\sum_{k=1}^{+\infty}\frac{(-1)^k}{(4k^2-1)^2}\cos(2ks)\right) - A\cos(s), \tag{15}$$

$$H(v_0) \quad = \quad \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{+\infty}\frac{(-1)^j}{2j+1}\cos((2j+1)s), \tag{16}$$

where $A = \dfrac{-a_0}{\pi\omega_0^2}\left(1 - 2\displaystyle\sum_{k=1}^{+\infty}\frac{(-1)^k}{(4k^2-1)^2}\right)$.
To remove secular term of order one in (13), it suffices to take α_2 such that:

$$0 \quad = \quad \int_0^{2\pi}[H(v_0(s))v_1(s) + \alpha_{\mathbf{2}} v_0''(s) + \alpha_1 v_1''(s)] \centerdot v_0(s)ds \tag{17}$$

For Duffing equation, see [15, 16, 17], the source term involves only few complex exponentials and the calculus of α_2 is explicit. For general smooth source term, Fourier coefficients decay very fast. Here, we have an infinite set of frequencies for v_1 and $H(v_0)$, with only a small algebraic rate of decay for Fourier coefficients. So, numerical computations need to compute a large number of Fourier coefficients. For our first simple example, we can compute explicitly α_2. After lengthy and tedious computations involving numerical series, we have from (17) and (14), (15), (16) to evaluate exactly the following numerical series:

$$\begin{aligned}
\alpha_2 \quad &= \quad -\frac{2}{(\pi\omega_0)^2}\left(1 - \frac{1}{9} + \sum_{j=1}^{+\infty}(2j+1)^{-1}(4j^2-1)^{-2} - (4(j+1)^2-1)^{-2}\right)\\
&= \quad -3(4\omega_0)^{-2},
\end{aligned}$$

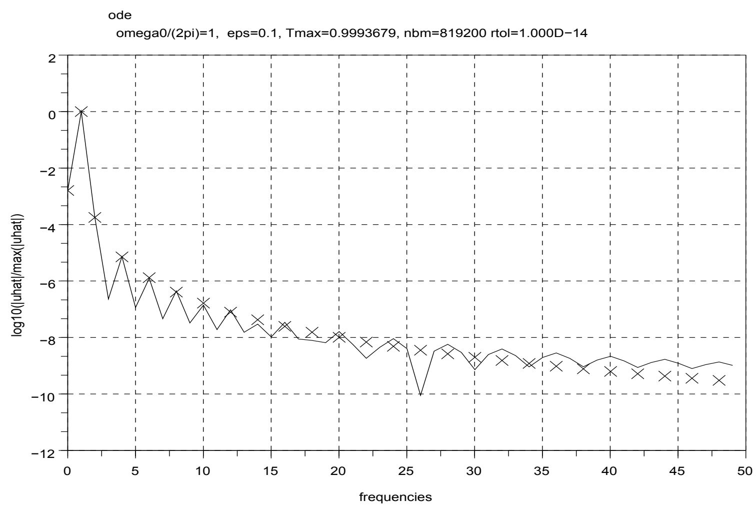

Figure 2: v_1, $\log_{10}\left(\dfrac{|\widehat{u}_\varepsilon|}{\max|\widehat{u}_\varepsilon|}\right)$

thus $\omega_2 = -(2\omega_0)^{-3}$ as we have yet obtained in (2).

When $a_0 = 1$ we obtain in figure 2 first modes of the infinite Fourier spectra for

$$v_0(\omega_\varepsilon t) + \varepsilon v_1(\omega_\varepsilon t) \quad \simeq \quad u_\varepsilon(t):$$

Indeed we have for negative or positive a_0 the following result.

Proposition 2.1 *Let u_ε be the solution of (1) with the initial data:*

$$u_\varepsilon(0) = a_0 + \varepsilon a_1, \qquad \dot{u}_\varepsilon(0) = 0,$$

then there exists $\gamma > 0$, such that, for all $t < T_\varepsilon = \gamma\varepsilon^{-1}$, we have the following expansion:

$$\begin{aligned}
u_\varepsilon(t) &= v_0(\omega_\varepsilon t) + \varepsilon v_1(\omega_\varepsilon t) + \mathcal{O}(\varepsilon^2), \\
v_0(\omega_\varepsilon t) &= a_0 \cos(\omega_\varepsilon t), \\
v_1(\omega_\varepsilon t) &= \frac{-|a_0|}{\pi\omega_0^2}\left(1 - 2\sum_{k=1}^{+\infty} \frac{(-1)^k}{(4k^2-1)^2}\cos(2k\omega_\varepsilon t)\right) + (a_1 - A)\cos(\omega_\varepsilon t), \\
\omega_\varepsilon &= \omega_0 + \frac{1}{4\omega_0}\varepsilon - \frac{1}{(2\omega_0)^3}\varepsilon^2,
\end{aligned}$$

where $A = \dfrac{-|a_0|}{\pi\omega_0^2}\left(1 - 2\displaystyle\sum_{k=1}^{+\infty} \frac{(-1)^k}{(4k^2-1)^2}\right)$.

In the Proposition 2.1, with the method of strained coordinates, we recover an asymptotic expansion for the exact angular frequency $\omega(\varepsilon) = \omega_0 + \omega_1\varepsilon + \omega_2\varepsilon^2 + \mathcal{O}(\varepsilon^3)$ and for the exact solution $u_\varepsilon(t)$. The term v_1 is explicitly given by its Fourier expansion. Notice also that ω_2 is not so easy to compute. It needs to compute a numerical series.
The technical proof of the Proposition 2.1 is postponed to the appendix.
Examples from Proposition 2.1 have angular frequency independent of the amplitude. Equation (1) is homogeneous . Indeed, it is a special case, as we can see in the non homogeneous following cases. In these cases, we assume that the spring is either not in contact with the mass at rest ($b > 0$) or with a prestress at rest ($b < 0$).

Proposition 2.2 (Nonlinear dependence of angular frequency)
Let b be a real number and let u_ε be the solution of:

$$\ddot{u} + \omega_0^2 u + \varepsilon a(u-b)_+ = 0, \qquad u_\varepsilon(0) = a_0 + \varepsilon a_1,\ \dot{u}_\varepsilon(0) = 0.$$

If $|a_0| > |b|$ then there exists $\gamma > 0$, such that, for $t \in [0, T_\varepsilon]$, with $T_\varepsilon = \dfrac{\gamma}{\varepsilon}$, we have the following expansion in $C^2([0, T_\varepsilon], \mathbb{R})$:

$$\begin{aligned}
u_\varepsilon(t) &= v_0(\omega_\varepsilon t) + \varepsilon v_1(\omega_\varepsilon t) + \mathcal{O}(\varepsilon^2) \qquad \textit{with}\\
v_0(s) &= a_0\cos(s),\\
v_1(s) &= \sum_{k=1}^{+\infty} d_k \cos(ks),
\end{aligned}$$

$$\begin{aligned}
d_0 &= -\frac{a|a_0|}{\pi\omega_0^2}\left(\sin(\beta) - \frac{b}{|a_0|}\beta\right) \qquad \textit{where } \beta = \arccos\left(\frac{b}{a_0}\right) \in [0,\pi],\\
d_k &= \frac{-a|a_0|}{\pi\omega_0^2(1-k^2)}\left(\frac{\sin((k+1)\beta)}{k+1} + \frac{\sin((k-1)\beta)}{k-1} - \frac{2b\sin(k\beta)}{|a_0|k}\right),\ k \geq 2,\\
d_1 &= a_1 - A \qquad \textit{where } A = \sum_{k\neq 1} d_k,\\
\omega_\varepsilon &= \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2,\\
\omega_1 &= \frac{a}{2\pi\omega_0}\left(\frac{\sin(2\beta)}{2} + \beta - \frac{2b\sin(\beta)}{|a_0|}\right),\\
\omega_2 &= \frac{\omega_1 d_1}{a_0} - \frac{a}{\omega_0\pi a_0}\int_0^\pi H(a_0\cos(s)-b)v_1(s)\cos(s)ds.
\end{aligned}$$

Notice that if $|a_0| < |b|$, there is no interaction with the weak unilateral spring. Thus the linearized solution is the exact solution.
Proof : There are two similar cases, a_0 positive or negative.
First case: assume $a_0 > 0$. With the previous notations, the method of strained coordinates yields the following equations:

$$\begin{aligned}
v_0'' + v_0 &= 0, \quad v_0(0) = a_0,\ \dot{v}_0(0) = 0 \quad \text{so } v_0(s) = a_0\cos(s),\\
-\alpha_0(v_1'' + v_1) &= a(v_0-b)_+ - \alpha_1 v_0 = aa_0(\cos(s) - b/a_0)_+ - \alpha_1 a_0\cos(s),\\
-\alpha_0(r_\varepsilon'' + r_\varepsilon) &= aH(v_0-b)v_1 - \alpha_2 v_0 - \alpha_1 v_1 + R_\varepsilon.
\end{aligned}$$

After some computations the Fourier expansion of $(\cos(s)-c)_+$ for $|c| < 1$ is:

$$(\cos(s)-c)_+ = \sum_{k=0}^{+\infty} c_k\cos(ks), \tag{18}$$

$$c_0 = \frac{\sin(\beta) - c\beta}{\pi} \qquad \text{where } \beta = \arccos(c) \in [0,\pi], \tag{19}$$

$$c_1 = \frac{1}{\pi}\left(\frac{\sin(2\beta)}{2} + \beta - 2c\sin(\beta)\right), \tag{20}$$

$$c_k = \frac{1}{\pi}\left(\frac{\sin((k+1)\beta)}{k+1} + \frac{\sin((k-1)\beta)}{k-1} - \frac{2c\sin(k\beta)}{k}\right),\ k \geq 2. \tag{21}$$

The non secular condition $\displaystyle\int_0^\pi (a(v_0-b)_+ - \alpha_1 v_0)\cos(s)ds = 0$, with $c = b/a_0$ yields $\alpha_1 = ac_1$.
Now, we can compute $\omega_1 = \alpha_1/(2\omega_0)$ and v_1 with a cosines expansion: $v_1(s) = \displaystyle\sum_k d_k\cos(ks)$

with $d_k = -\dfrac{aa_0}{\alpha_0}\dfrac{c_k}{1-k^2}$ for $k \neq 1$. The coefficient d_1 is then obtained with the initial data $v_1(0) = a_1$, $\dot{v}_1(0) = 0$.
α_2, is obtained with the non secular condition for r_ε:
$0 = \dfrac{1}{\pi}\displaystyle\int_0^\pi (aH(v_0 - b)v_1 - \alpha_2 v_0 - \alpha_1 v_1)\cos(s)ds$. This condition is rewritten as follow
$\alpha_2 = \dfrac{2\omega_0\omega_1 d_1}{a_0} - \dfrac{2a}{\pi a_0}\displaystyle\int_0^\pi H(a_0\cos(s) - b)v_1(s)\cos(s)ds$, which gives ω_2 since $\omega_2 = \frac{\alpha_2 - \omega_1^2}{2\omega_0}$.
Second case: when $a_0 = -|a_0| < 0$, by a similar way, we obtain a similar expansion, except that $(v_0(s) - b)_+ = |a_0|(-\cos(s) - b/a_0)_+$. And we only need the Fourier expansion of $(-\cos(s) - c)_+ = \sum_k \tilde{c}_k \cos(ks)$,

$$\begin{aligned}
\tilde{c}_0 &= -\frac{\sin(\beta) + c\beta}{\pi}, \\
\tilde{c}_1 &= -\frac{1}{\pi}\left(\frac{\sin(2\beta)}{2} + \beta + 2c\sin(\beta)\right), \\
\tilde{c}_k &= -\frac{1}{\pi}\left(\frac{\sin((k+1)\beta)}{k+1} + \frac{\sin((k-1)\beta)}{k-1} + \frac{2c\sin(k\beta)}{k}\right), \ k \geq 2.
\end{aligned}$$

□

When $|a_0| = |b|$, we have another asymptotic expansion only valid for time of order $\frac{1}{\sqrt{\varepsilon}}$ when the unilateral spring interacts with the mass.

Proposition 2.3 (The critical case)
Let b be a real number, $b \neq 0$, and consider, the solution u_ε of:

$$\ddot{u} + \omega_0^2 u + \varepsilon a(u - b)_+ = 0, \qquad u_\varepsilon(0) = a_0 + \varepsilon a_1, \ \dot{u}_\varepsilon(0) = 0.$$

If $|a_0| = |b|$ then we have

$$u_\varepsilon(t) = (a_0 + \varepsilon a_1)\cos(\omega_0 t) + \mathcal{O}(\varepsilon^2),$$

for $t \leq T_\varepsilon = \begin{cases} \dfrac{\gamma}{\sqrt{\varepsilon}} & \text{if} \quad |a_0 + \varepsilon a_1| > |b| \quad \text{where } \gamma > 0, \\ +\infty & \text{else.} \end{cases}$

The method of strained coordinates gives us the *linear* approximation for $u_\varepsilon(t)$, with $s = t$, i.e. $\omega_\varepsilon = 1$. If $|u_\varepsilon(0)| < |b|$, the exact solution is the solution of the linear problem $\ddot{u} + \omega_0^2 u = 0$. Otherwise, if $|u_\varepsilon(0)| > |b|$, since, $|b|$ is the maximum of $v_0(s) = a_0\cos(s)$, a new phenomenon appears, during each period, $|u_\varepsilon(t)| > |b|$ on interval of time of order $\sqrt{\varepsilon}$ instead of ε. Then T_ε is smaller than in Proposition 2.1.
To explain this phenomenon, we give precise estimates of the remainder when we expand $(v_0 + \varepsilon v_1 + \varepsilon^2 r_\varepsilon)_+$ in the next section, see Lemmas 3.1 and 3.2 below.

3 Expansion of $(u + \varepsilon v)_+$

We give some useful lemmas to perform asymptotic expansions and to estimate precisely the remainder for the piecewise linear map $u \to u_+ = \max(0, u)$.

Lemma 3.1 [Asymptotic expansion for $(u + \varepsilon v)_+$]
Let be $T > 0$, $M > 0$, u, v two real valued functions defined on $I = [0, T]$,

$$J_\varepsilon = \{t \in I, |u(t)| \leq \varepsilon M\},$$

$\mu_\varepsilon(T)$ the measure of the set J_ε and H be the Heaviside step function, then

$$(u + \varepsilon v)_+ = (u)_+ + \varepsilon H(u)v + \varepsilon\chi_\varepsilon(u, v), \quad \text{with} \quad H(u) = \begin{cases} 1 & \text{if } u > 0 \\ 0 & \text{elsewhere} \end{cases},$$

and $\chi_\varepsilon(u,v)$ is a non negative piecewise linear function and 1-Lipschitz with respect to v, which satisfies for all ε,
if $|v(t)| \leq M$ for any $t \in I$:

$$|\chi_\varepsilon(u,v)| \leq |v| \leq M, \qquad \int_0^T |\chi_\varepsilon(u(t),v(t))|\,dt \leq M\mu_\varepsilon(T). \tag{22}$$

The point in inequality (22) is the remainder $\varepsilon\chi_\varepsilon$ is only of order ε in L^∞ but of order $\varepsilon\mu_\varepsilon$ in L^1. In general, μ_ε is not better than a constant, take for instance $u \equiv 0$. Fortunately, it is proved below that μ_ε is often of order ε, and for some critical cases of order $\sqrt{\varepsilon}$.
Proof : Equality (22) defines χ_ε and can be rewritten as follow:

$$\chi_\varepsilon(u,v) \quad = \quad \frac{(u+\varepsilon v)_+ - u_+ - \varepsilon H(u)v}{\varepsilon}. \tag{23}$$

So, χ_ε is non negative since $u \to u_+$ is a convex function. We also easily see that the map $(u,v) \to \chi_\varepsilon(u,v)$ is piecewise linear, continuous except on the line $u = 0$ where χ_ε has a jump $-v$. This jump comes from the Heaviside step function. An explicit computations gives us the simple and useful formula:

$$0 \leq \varepsilon\chi_\varepsilon(u,v) \quad = \quad \begin{cases} |u+\varepsilon v| & \text{if } |u+\varepsilon v| < |\varepsilon v| \\ 0 & \text{elsewhere} \end{cases}. \tag{24}$$

We then have immediately $0 \leq \chi_\varepsilon(u,v) \leq |v|$. Let u be fixed, then $v \to \chi_\varepsilon(u,v)$ is one Lipschitz with respect to v. Furthermore, the support of χ_ε is included in J_ε, which concludes the proof. □

We now investigate the size of $\mu_\varepsilon(T)$, see [3, 10] for similar results about $\mu_\varepsilon(T)$ and other applications.
With notations from Lemma 3.1 we have.

Lemma 3.2 (Order of $\mu_\varepsilon(T)$) *Let u be a smooth periodic function, M be a positive constant and $\mu_\varepsilon(T)$ the measure of the set $\{t \in I, |u(t)| \leq \varepsilon M\}$.*
If u has only simple roots on $I = [0,T]$ then for some positive C,

$$\mu_\varepsilon(T) \quad \leq \quad C\varepsilon \times T.$$

More generally, if u has also double roots then

$$\mu_\varepsilon(T) \quad \leq \quad C\sqrt{\varepsilon} \times T.$$

The measure of such set J_ε implies many applications in averaging lemmas, for a characterization of μ_ε in a multidimensional framework see [3, 10].
Notice that any non zero solution $u(t)$ of any linear homogeneous second order ordinary differential equation has always simple zeros, thus for any constant c the map $t \to u(t) - c$ has at most double roots.

Proof : First assume u only has simple roots on a period $[0,P]$, and let $Z = \{t_0 \in [0,P], u(t_0) = 0\}$. The set Z is discrete since u has only simple roots which implies that roots of u are isolated. Thus Z is a finite subset of $[0,P]$: $Z = \{t_1, t_2, \cdots, t_N\}$. We can choose an open neighborhood V_j of each t_j such that u is a diffeomorphism on V_j with derivative $|\dot{u}| > |\dot{u}(t_j)|/2$. On the compact set $K = [0,P] - \cup V_j$, u never vanishes, then $\min\limits_{t\in K} |u(t)| = \varepsilon_0 > 0$. Thus, we have for all $\varepsilon M < \varepsilon_0$, the length of J_ε in V_j is $|V_j \cap J_\varepsilon| \leq \dfrac{4\varepsilon M}{|\dot{u}(t_j)|}$.
As μ_ε is additive ($\mu_\varepsilon(P+t) = \mu_\varepsilon(P) + \mu_\varepsilon(t)$), its growth is linear. Thus, for the case with simple roots, we get $\mu_\varepsilon(T) = \mathcal{O}(\varepsilon T)$.
For the general case with double roots, on each small neighborhood of t_j: V_j, we have with a Taylor expansion, $|u(t_j+s)| \geq d_j|s|^l$, with $1 \leq l \leq 2$, $d_j > 0$, so, $|V_j \cap J_\varepsilon| \leq 2(\varepsilon M/d_j)^{1/l}$, then $\mu_\varepsilon(P) = \mathcal{O}(\sqrt{\varepsilon})$,which is enough to conclude the proof. □

4 Several degrees of freedom

Now, we investigate the case with N mass. We use, the method of strained coordinates in three cases. We present the formal computations for each expansion. The mathematical proofs are postponed in the Appendix.
In subsection 4.1, the initial condition is near an eigenvector such that the approximate solution stays periodic. We gives such initial condition near an eigenvector in subsection 4.2 to get an approximate nonlinear normal mode up to the order ε^2. Finally, in subsection 4.4, all modes are excited. An extension of the method of strained coordinates is still possible but only at the first order with less accuracy.
The system studied is the following:
$M\ddot{U} + KU + \varepsilon(AU - B)_+ = 0$, where, for each component,
$[(AU-B)_+]_k = \left(\sum_{j=1}^{N} a_{kj}u_j - b_k\right)_+$, M is a diagonal $N \times N$ mass matrix with positive terms on the diagonal, K is the stiffness matrix which is symmetric definite positive. It is also possible to add many terms $\varepsilon(AU - B)_+$ modeling small defects. For a such system, endowed with a natural convex energy for the linearized part, we can control the ε-Lipschitz last term for ε small enough up to large time, so for $\varepsilon << 1$ the solutions remain bounded for time of the order ε^{-1}.
To simplify the exposition, we deal with the following diagonalized system for the linear part, keeping the same notations, except for the positive diagonal matrix Λ:

$$\ddot{U} + \Lambda^2 U \quad = \quad -\varepsilon(AU - B)_+. \tag{25}$$

In fact, with a linear change of variables, we have in the right hand side of the previous equation (25) a linear combinations of terms $\varepsilon(AU - B)_+$.

4.1 Initial condition near an eigenvector, second order approximation

For the system (25), we take an initial condition near an eigenmode of the linearized system denoted for instance by index 1 .

$$\left\{ \begin{array}{rcl} u_1^\varepsilon(0) & = & a_0 + \varepsilon a_1, \quad \dot{u}_1^\varepsilon(0) = 0, \\ u_k^\varepsilon(0) & = & 0 \ \ + \varepsilon a_k, \quad \dot{u}_k^\varepsilon(0) = 0, \quad for\ k \neq 1. \end{array} \right.$$

We impose $a_2, \cdots, a_N$ later to have a periodic approximation, but a_1 is a free constant as a_0. It is a key point to apply the method of strained coordinates.
We use the same time $s = \omega_\varepsilon t$ for each component and the following notations.

$$\begin{array}{rclrcl} \omega_\varepsilon & = & \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2, & \omega_0 & = & \lambda_1, \\ (\omega_\varepsilon)^2 & = & \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \mathrm{O}(\varepsilon^3), & \alpha_0 & = & \omega_0^2 = \lambda_1^2, \\ \alpha_1 & = & 2\omega_0\omega_1, & \alpha_2 & = & \omega_1^2 + 2\omega_0\omega_2, \\ u_j^\varepsilon(t) & = & v_j^\varepsilon(s) = v_j^0(s) + \varepsilon v_j^1(s) + \varepsilon^2 r_j^\varepsilon(s), & j & = & 1, \cdots, N. \end{array}$$

Replacing, this ansatz in the System (25) we have in variable s,

$$(\omega_\varepsilon)^2 (v_k^\varepsilon)'' + \lambda_k^2 v_k^\varepsilon \quad = \quad -\varepsilon \left(\sum_{j=1}^{N} a_{kj} v_j^\varepsilon(s) - b_k \right)_+ ,$$

and then performing the expansion for all $k \in \{1, \cdots, N\}$,

$$\begin{aligned} L_k v_k^0 &= \alpha_0 (v_k^0)'' + \lambda_k^2 v_k^0 = 0, \\ -L_k v_k^1 &= \left(\sum_{j=1}^N a_{kj} v_j^0 - b_k \right)_+ + \alpha_1 (v_k^0)'', \\ -L_k r_k^\varepsilon &= H\left(\sum_{j=1}^N a_{kj} v_j^0 - b_k \right) \left(\sum_{j=1}^N a_{kj} v_j^1 \right) + \alpha_2 (v_k^0)'' + \alpha_1 (v_k^1)'' + R_k^\varepsilon. \end{aligned}$$

First we have $v_1^0(s) = a_0 \cos(s)$.
Equations for v_k^0, for all $k \neq 1$, with zero initial data give us $v_k^0 = 0$.
In equation for v_1^1, we remove the secular term in the right hand side,

$$-\alpha_0((v_1^1)'' + v_1^1) = \left(a_{11} v_1^0 - b_1\right)_+ + \alpha_1 (v_1^0)'' = r.h.s. \qquad v_1^1(0) = a_1, \; (v_1^1)'(0) = 0.$$

The orthogonality of the $r.h.s$ with $\cos(s)$ leads to define α_1. For instance, if $b_1 = 0$, we have as in Proposition 2.2, $\alpha_1 = \frac{a_{11}}{2}$ and $\omega_1 = \frac{a_{11}}{4\lambda_1}$.
Now, α_1 is fixed, so v_1^1 is a well defined even 2π periodic function.
Then, for $k \neq 1$, we can compute v_k^1 with the simplified equation,

$$\alpha_0 (v_k^1)'' + \lambda_k^2 v_k^1 = -\left(a_{k1} v_1^0 - b_k\right)_+ . \tag{26}$$

Let ϕ_k^1 be the unique 2π periodic solution of equation (26). Such function exists and is unique if $\lambda_k \notin \lambda_1 \mathbb{Z}$. Furthermore ϕ_k^1 is an even function as the right hand side of equation (26). Let v_k^1 be ϕ_k^1, i.e. $v_k^1(0) = a_k = \phi_k^1(0)$ and $(v_k^1)'(0) = 0$ for all $k \neq 1$.
The term r_1^ε, with null initial data, has a simplified equation since $v_k^0 \equiv 0$ for all $k \neq 1$,

$$-L_1 r_1^\varepsilon = H\left(a_{11} v_1^0 - b_1\right) \left(\sum_{j=1}^N a_{1j} v_j^1 \right) + \alpha_2 (v_1^0)'' + \alpha_1 (v_1^1)'' + R_1^\varepsilon.$$

We now can compute numerically α_2 to avoid secular term in the right hand side, R_1^ε excepted, with the following condition,

$$0 = \int_0^\pi \left[H\left(a_{11} v_1^0 - b_1\right) \left(\sum_{k=1}^N a_{1k} v_k^1 \right) + \alpha_2 (v_1^0)'' + \alpha_1 (v_1^1)'' \right] \cdot \cos(s) ds.$$

Rewriting this condition, we obtain an equation for α_2 in Theorem 4.1 below, which gives ω_2 as in the Proposition 2.2.
For each $k \neq 1$, λ_k and λ_1 are $\mathbb{Z}$ independent. So r_k^ε stays bounded for large time since there is no resonance of the order one at the first order in equations, $-L_k r_k^\varepsilon = (\cdots) + R_k^\varepsilon$. This is the technical part of the proof to validate rigorously and to find the time of validity of such asymptotic expansion. The complete proof to bound $(r_1^\varepsilon, \cdots, r_N^\varepsilon)$ for large time is to be found in the Appendix.
We now state our result with previous notations.

Theorem 4.1 *The following expansion is valid on $(0, T_\varepsilon)$, with $T_\varepsilon \to +\infty$ when $\varepsilon \to 0$ under assumption λ_k and λ_1 are $\mathbb{Z}$ independent for each $k \neq 1$,*

$$\left\{ \begin{array}{rclclcl} u_1^\varepsilon(t) &=& v_1^0(\omega_\varepsilon t) &+& \varepsilon v_1^1(\omega_\varepsilon t) &+& \mathcal{O}(\varepsilon^2), \\ u_k^\varepsilon(t) &=& 0 &+& \varepsilon v_k^1(\omega_\varepsilon t) &+& \mathcal{O}(\varepsilon^2), \quad k \neq 1, \end{array} \right. \tag{27}$$

where $v_1^0(s)$, α_1, ω_1, $v_1^1(s)$, $v_k^1(s)$, ϕ_k^1, α_2, ω_2 are defined successively as follows, with differential operators $L_k = \lambda_1^2 \dfrac{d^2}{ds^2} + \lambda_k^2$,

$$
\begin{aligned}
v_1^0(s) &= a_0 \cos(s), && (28)\\
\alpha_1 &= \frac{2}{a_0 \pi} \int_0^{\pi} (a_{11} v_1^0(s) - b_1)_+ \cos(s) ds, && (29)\\
\omega_1 &= \frac{\alpha_1}{2\omega_0}, && (30)\\
-L_1 v_1^1 &= \left(a_{11} v_1^0 - b_1\right)_+ + \alpha_1 (v_1^0)'', \quad v_1^1(0) = a_1,\ (v_1^1)'(0) = 0, && (31)\\
-L_k v_k^1 &= \left(a_{k1} v_1^0 - b_k\right)_+, \quad v_k^1(0) = a_k := \phi_k^1(0),\ (v_k^1)'(0) = 0,\ \textit{for } k \neq 1, && (32)\\
\phi_k^1 &\ \textit{is the unique } 2\pi \textit{ periodic solution of (32) for all } k \neq 1, && (33)\\
\alpha_2 &= \left(\frac{2}{a_0 \pi} \int_0^{\pi} \left\{ H(a_{11} v_1^0(s) - b_1) \left(a_{11} v_1^1 + \sum_{k \neq 1}^{N} a_{1k} \phi_k^1 \right) \right\} \cos(s) ds \right) - \alpha_1, && (34)\\
\omega_2 &= \frac{\alpha_2 - \omega_1^2}{2\omega_0}. && (35)
\end{aligned}
$$

Furthermore, if $(a_{j1} v_1^0 - b_j)$ has got only simple roots for all $j = 1, \cdots, N$,
then T_ε is of the order ε^{-1},
else T_ε is of the order $\varepsilon^{-1/2}$.

In the theorem ϕ_k^1 is classically obtained by a Fourier series. We give some indication of its initial condition in the next subsection 4.2.

4.2 Approximate non linear normal mode

The special initial conditions of the previous subsection can be explicited in order to find a solution where all the components are in phase at the same frequency. Indeed we shall obtain an approximate curve of initial conditions for which the solution is periodic up to the order ε for a time of the order $\dfrac{1}{\varepsilon}$: this is up to the approximation a non linear normal mode in the sense of Rosenberg [22]; see [9] for a computation of non linear normal mode with unilateral contact and [14] for a synthesis on non linear normal modes.

Theorem 4.2 *For the system (25), we take an initial condition close to an eigenmode of the linearized system denoted for instance by index 1 .*

$$
\left\{
\begin{array}{lcl}
u_1^\varepsilon(0) &=& a_0 + \varepsilon a_1, \quad \dot{u}_1^\varepsilon(0) = 0,\\
u_k^\varepsilon(0) &=& 0 + \varepsilon a_k, \quad \dot{u}_k^\varepsilon(0) = 0, \quad for\ k \neq 1.
\end{array}
\right.
$$

the results of the previous theorem are still valid for $T_\varepsilon = \dfrac{\gamma}{\varepsilon}$ but now we are looking for $a_k = v_k^1(0) =$ in (32).
Moreover for the cases detailed below, the solution is periodic with angular frequency ω_ε up to the order ε and for $0 \leq t \leq T_\varepsilon$

1. *assume $b_k = 0$, for $k \neq 1$ with the initial condition defined with*

$$a_k = \frac{a_{k1} a_0}{2(\lambda_1^2 - \lambda_k^2)} - \frac{|a_{k1} a_0|}{\lambda_k^2 \pi} + \frac{a_{k1} a_0}{2} \sum_{l=1}^{+\infty} \frac{(-1)^l}{(4l^2 \lambda_1^2 - \lambda_k^2)(4l^2 - 1)} \tag{36}$$

2. *assume $0 < \frac{b_k}{|a_{k1} a_0|} < 1$, and $a_0 a_{k1} < 0$ for $k \neq 1$ then for the initial condition defined with*

$$a_k = a_0 a_{k1} \left[\frac{1}{\lambda_1^2 - \lambda_k^2} - \frac{b_k}{\lambda_k^2 a_{k1} a_0} - \sum_{l=1}^{+\infty} \frac{c_l}{l^2 \lambda_1^2 - \lambda_k^2} \right] \tag{37}$$

where c_l are defined in (18) *with* $c = -\frac{b_k}{a_{k1}a_0}$

3. *assume* $0 < \frac{|b_k|}{|a_{k1}a_0|} < 1$, *and* $a_0 a_{k1} > 0$ *for* $k \neq 1$ *then for the initial condition defined with*

$$a_k = a_0 a_{k1} \left[\sum_{l=1}^{+\infty} \frac{c_l}{l^2\lambda_1^2 - \lambda_k^2} \right] \tag{38}$$

where c_l are defined in (18) *with* $c = \frac{b_k}{a_{k1}a_0}$

Remark 4.1 *The other cases are less interesting but may solved similarly.*

Proof : The principle of the proof is simple; v_k^1 is solution of the differential equation (32) with $v_k^1(0) = a_k$ and a_k has to be determined in order that the function v_k^1 has an angular frequency equal to one. It is elementary that the solution of (32) is

$$v_k^1 = A\cos\left(\frac{\lambda_k}{\lambda_1}s\right) + B\sin\left(\frac{\lambda_k}{\lambda_1}s\right) + \phi_k^1(s) \tag{39}$$

where ϕ_k^1 is a particular solution associated to the right hand side which is of angular frequency equal to 1; note that $B = 0$ as the initial velocity is null; we can get a function of angular frequency equal to 1 by setting $a_k = \phi_k(0)$; this condition may be written explicitly with formulas (18) which provides the expansion in Fourier series; formulas (36), (37), (38) are then derived easily.

1. for $b_k = 0$ for $k \neq 1$, (32) is written:

$$-L_k v_k^1 = a_{k1}a_0 \frac{cos(s)}{2} + |a_{k1}a_0| \frac{cos(s)}{2} \tag{40}$$

we use formula (14) to get the particular solution

$$\phi_k^1 = a_{k1}a_0 \frac{cos(s)}{2(\lambda_1^2 - \lambda_k^2)} - \frac{|a_{k1}a_0|}{\lambda_k^2 \pi} + \frac{|a_{k1}a_0|}{2} \sum_{l=1}^{+\infty} \frac{(-1)^l}{4l^2\lambda_1^2 - \lambda_k^2 1} \frac{cos(2ls)}{4l^2 - 1} \tag{41}$$

from which (36) is deduced.

2. for $0 < \frac{b_k}{|a_{k1}a_0|} < 1$, and $a_0 a_{k1} < 0$ for $k \neq 1$ (32) is written:

$$-L_k v_k^1 = -a_{k1}a_0 \left(-cos(s) + \frac{b_k}{a_{k1}a_0}\right)_+ \quad \text{or} \tag{42}$$

$$-L_k v_k^1 = -a_{k1}a_0 \left[\left(-cos(s) + \frac{b_k}{a_{k1}a_0}\right) + \left(cos(s) - \frac{b_k}{a_{k1}a_0}\right)_+\right] \quad \text{we use (22) to obtain} \tag{43}$$

$$\phi_k^1 = -a_{k1}a_0 \left[\left(\frac{-cos(s)}{2(\lambda_1^2 - \lambda_k^2)} - \frac{b_k}{\lambda_k^2 a_{k1}a_0}\right) + \sum_{l=1}^{+\infty} \frac{c_l cos(ls)}{l^2\lambda_1^2 - \lambda_k^2} \right] \quad \text{where } c_l \text{ is defined with} \tag{44}$$

$$\beta = arccos\left(\frac{b_k}{a_{k1}a_0}\right) \text{ which defines } c_l \text{ in (22) with } c = \frac{-b_k}{a_{k1}a_0} \tag{45}$$

3. for $0 < \frac{b_k}{|a_{k1}a_0|} < 1$, and $a_0 a_{k1} > 0$ for $k \neq 1$ (32) is written:

$$-L_k v_k^1 = a_{k1}a_0 \left(cos(s) - \frac{b_k}{a_{k1}a_0}\right)_+ \quad \text{from which} \tag{46}$$

$$\phi_k^1 = a_0 a_{k1} \sum_{l=1}^{+\infty} \frac{c_l cos(ls)}{l^2\lambda_1^2 - \lambda_k^2} \quad \text{where } c_l \text{ is defined with} \tag{47}$$

$$\beta = arccos\left(\frac{b_k}{a_{k1}a_0}\right) \text{ which defines } c_l \text{ in (22) with } c = \frac{-b_k}{a_{k1}a_0} \tag{48}$$

$\square$

4.3 Numerical results of NNM

4.3.1 Using numerically Lindstedt-Poincaré expansions

Here we use the previous results and compute numerically a solution of system (25) using the approximation (27)

$$u^{\varepsilon}(t) = v^0(\omega_\varepsilon t) + \varepsilon v^1(\omega_\varepsilon t) + O(\varepsilon^2)$$

with the initial conditions of theorem 4.2. The first term v^0 is easy to obtain; for the second term v^1 an explicit formula is in principle possible using Fourier series such as for one degree of freedom but it is cumbersome so we choose to compute v^1 by solving numerically (32) with a step by step algorithm; we use as a black-box the routine ODE of SCILAB [25] to solve equations of theorem 4.1 after computing by numerical integration α_1. We show numerical results for a system of the type:

$$M\ddot{X} + KX + \varepsilon F(X) = 0 \tag{49}$$

we still denote λ_j^2 the eigenvalues and ϕ_j the eigenvectors of the usual generalized eigenvalue problem

$$K\phi_j - \lambda_j^2 M\phi_j = 0 \quad \text{with} \tag{50}$$

$$^t\phi_k M\phi_j = \delta_{kj} \tag{51}$$

We set:

$$X = \sum_j u_j\phi_j = \underline{\underline{\phi}}\underline{u} \tag{52}$$

In this basis, the system may be written componentwise:

$$\ddot{u_k} + \lambda_k^2 u_k +^t \phi_k F(\underline{\underline{\phi}}\underline{u}) = 0 \tag{53}$$

For a local non linearity in the system (49), written in the basis of the eigenvectors, we do not obtain a system precisely of the form (25); so we illustrate it with the non linearity:

$$F(X) = M\phi_1(X1 - \beta_1) = M\phi_1(\sum_j u_j\phi_{j1} - \beta_1)_+ \tag{54}$$

where we denote by ϕ_1 *any* normalized eigenvector of 50; then the system (49) is written:

$$\ddot{u_k} + \lambda_k^2 u_k + (\sum_k \delta_{k1}\phi_{j1}u_j - \delta_{k1}\beta_1)_+ \tag{55}$$

so that A and B of (25) are:

$$a_{kj} = \delta_{k1}\phi_{j1} \quad b_k = \delta_{k1}\beta_1$$

We find in figure 3 a numerical example of the Linstedt-Poincaré approximation for 5 degrees of freedom with $\varepsilon = 0.063$ and with an energy of 0.03002. The left figure shows the 5 components of the solution with respect to time; the right figure, the solution in the configuration space: absissa component 1 and ordinate components 2 to 5; these lines are rectilinear like in the linear case but the non symmetry may be particularly noticed on the smallest component which corresponds to the mode where the non linearity is active.

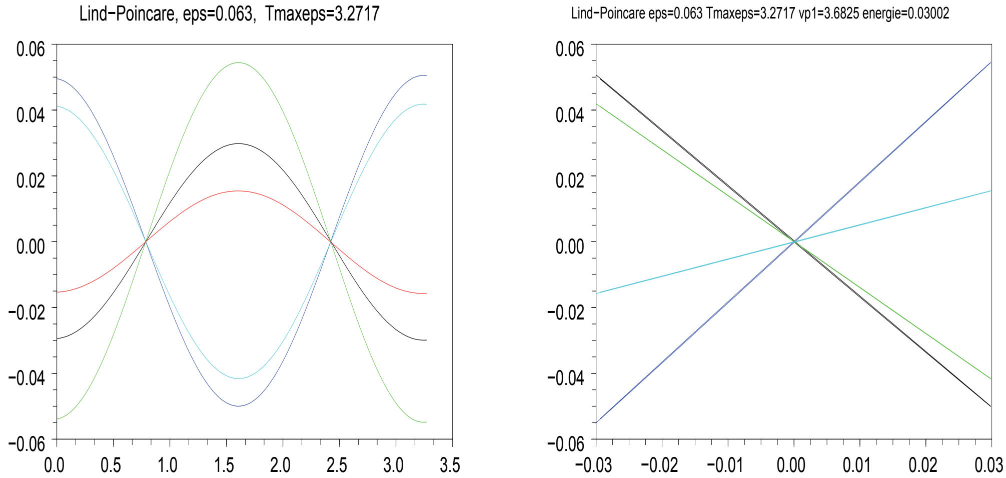

Figure 3: Lindstedt-Poincaré, energy=0.03, 5 dof; left: components with respect to time; right: in configuration space

4.3.2 Using optimization routines

We find also in figure 4.3.2 a numerical example with the same energy of 0.03002; it is computed with a purely numerical method described below. We notice that the solution is quite similar in both cases.

The numerical expansions of the previous subsection gives valid results for ε small enough; in many practical cases such as [7], ε may be quite large; in this case, it is natural to try to solve numerically the following equations with respect to the period T and the initial condition $X(0)$.

$$X(0) = X(T), \quad \dot{X}(0) = \dot{X}(T) \quad E(X) = e \tag{56}$$

In other words, we look for a periodic solution of prescribed energy; this last condition is to ensure to obtain an isolated local solution: the previous expansions show that in general, the period of the solution depends on its amplitude prescribed here by its energy. To try to solve these equations with a black-box routine for nonlinear equations such as "fsolve" routine of SCILAB [25](an implementation of a modification of Powell hybrid method which goes back to [20]) in general fails to converge. Even in case of convergence, we should address the question of link of this solution with normal modes of the linearized system.

So we prescribe that $e = c\varepsilon$ and for $\varepsilon \to 0$, the solution is tangent to a linear eigenmode. In the case where all the eigenvalues of the linear system are simple, we define N (the number of degrees of freedom) non linear normal modes for which, it is reasonable to conjecture that they correspond to isolated solutions of (56) at least for small ε if we enforce for example $\dot{X}(0) = 0$.

Algorithm This definition of the solution of (56) tangent to a prescribed linear eigenmode provides a simple way of numerical approximation: using a continuation method coupled with a routine for solving a system of non linear equations. Define:

$$\mathcal{F}(\varepsilon, X_0, T) = [X(T) - X_0, E(X) - c\varepsilon] \tag{57}$$

where X is a numerical solution of the differential system (58)

$$M\ddot{X} + KX + \varepsilon F(X) = 0 \tag{59}$$

$$X(0) = X_0, \quad \dot{X}(0) = X_1 \tag{60}$$

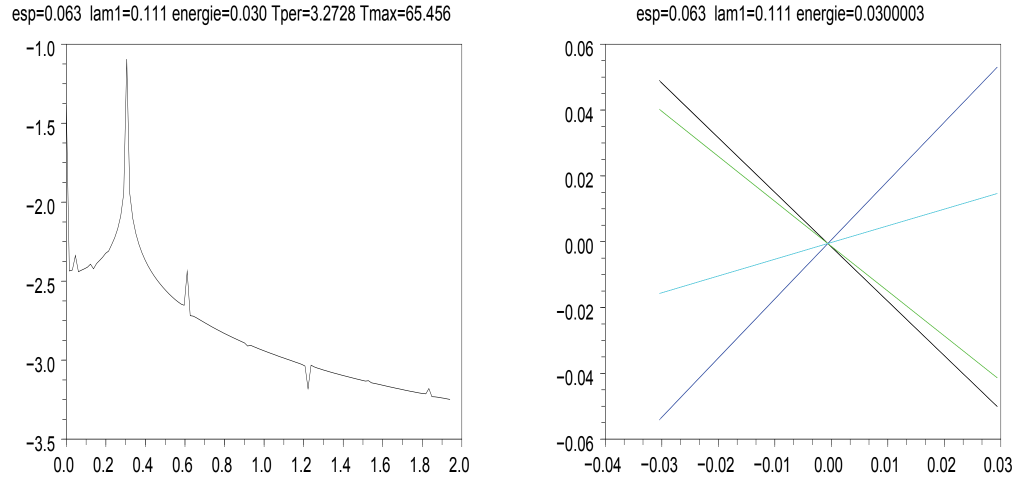

Figure 4: Continuation and Powell hybrid, energy=0.03, 5 dof; left:fourier transform ; right: in configuration space

choose a small initial value of ε and an increment δ
choose an eigenvector ϕ_j
$X_0(0) = A_\varepsilon \phi_j$, $X_1(0) = B_\varepsilon \lambda_j \phi_j$
with $E(X_0(0), X_1(0)) = c\varepsilon$

for iter=1:itermax

$$\varepsilon = \varepsilon + \delta$$

with $X_0(iter - 1)$ as a first approximation, solve for $X_0(iter)$,

$$\mathcal{F}(\varepsilon, X_0, T) = 0 \tag{61}$$

if $||\mathcal{F}(\varepsilon, X_0, T)|| > tolerance$ **then** $\varepsilon = \varepsilon - \delta, \quad \delta = \delta/2$
endif
endfor

This algorithm may be improved by using not only the solution associated to the previous value of ε to solve

$$\mathcal{F}(\varepsilon, X_0, T) = 0 \tag{62}$$

but also the derivative of the solution with respect to X_0, T.

Numerical results These results are obtained by solving the differential equation with a step by step numerical approximation of the routine *ode* of Scilab without prescribing the algorithm. As we are looking for a periodic solution, this numerical approximation may be certainly improved in precision and computing time by using an harmonic balance algorithm. In figure 5, 6, the same example with 5 degrees of freedom and energy equal 0.123 and 0.201 are displayed.

On the left of figure 5 we find the decimal logarithm of the absolute value of the Fourier transform of the solution; the Fourier transform is computed with the fast fourier transform with the routine fft of Scilab; we notice the frequency zero due to the non symmetry of the solution and multiples of the basic frequency; no other frequency appears; on the right the five components are plotted with respect to time; we still notice the non symmetry.

On the left of figure 6 we find the solution in the configuration space and on the right the five components are plotted with respect to time; we still notice the non symmetry.

In figure 7 we find results with 20 degrees of freedom, $\varepsilon = 0.272$ and energy of 0.129; the NNM is computed by starting with an eigenvector associated to the largest eigenvalue . We see on the left in the configuration space that the components are in phase and on the right, the Fourier transform shows zero frequencies and multiple of the basic frequency.

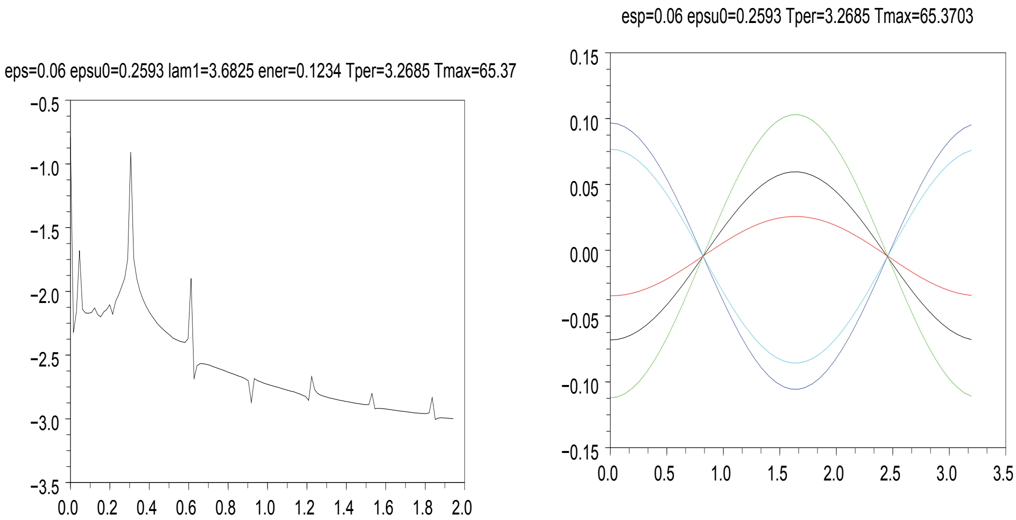

Figure 5: energy=0.123, 5 dof; left:fourier transform ; right: with respect to time

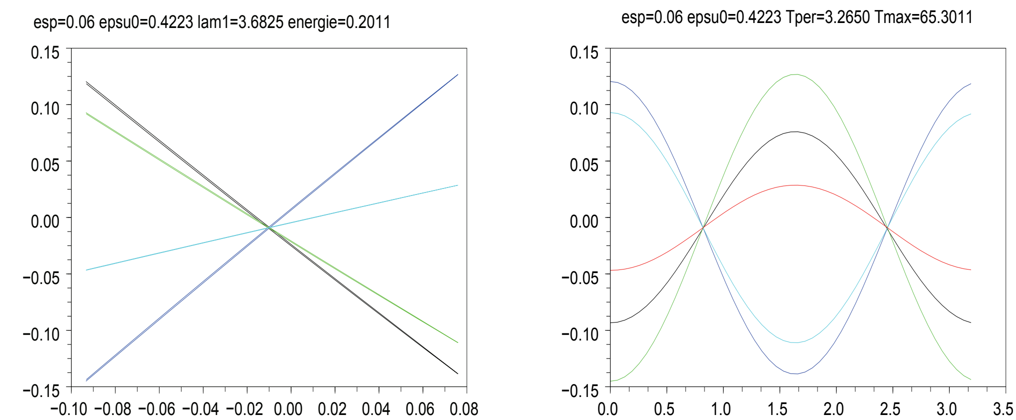

Figure 6: energy=0.2, 5 dof; left: configuration space; right:with respect to time

In figure 8 the energy is 0.29 and the NNM is computed by starting with an eigenvector associated to the smallest eigenvalue; we notice on the left, the solution in the configuration space: at zero each dof has a discontinuity in slope which is clear.

In figure 9, the shape of the NNM is displayed on the left for the NNM starting from the eigenvector associated to the smallest eigenvalue and on the right for the NNM starting from the second smallest eigenvalue. We notice that the shape is quite similar to the shape of the linear mode.

In figure 10, the NNM is computed by starting at an eigenvector associated to the second smalest eigenvalue. We notice that the first NNM is more asymmetric than the second one;

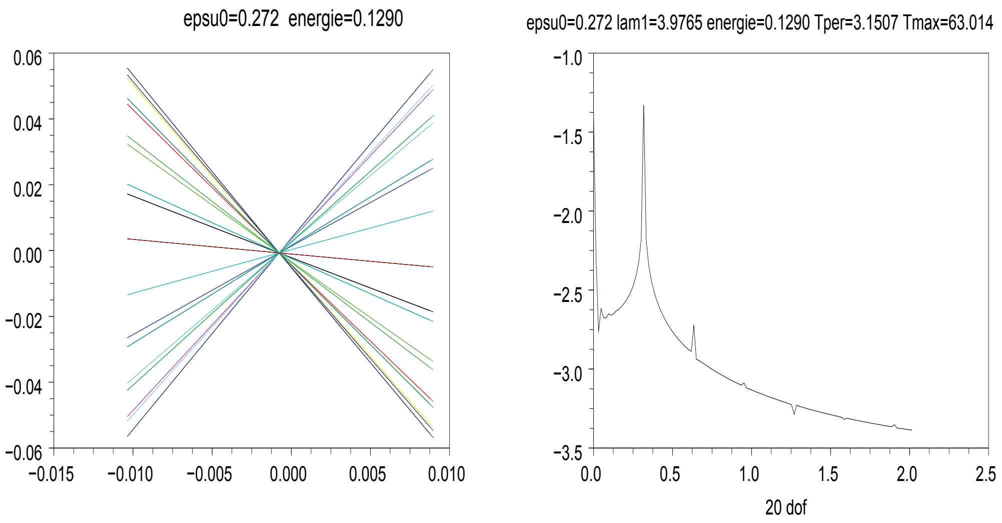

Figure 7: energy=0.129, 20 dof; left:in configuration space; right: fourier transform

on the left, the discontinuity of slope is smaller than for the first NNM.

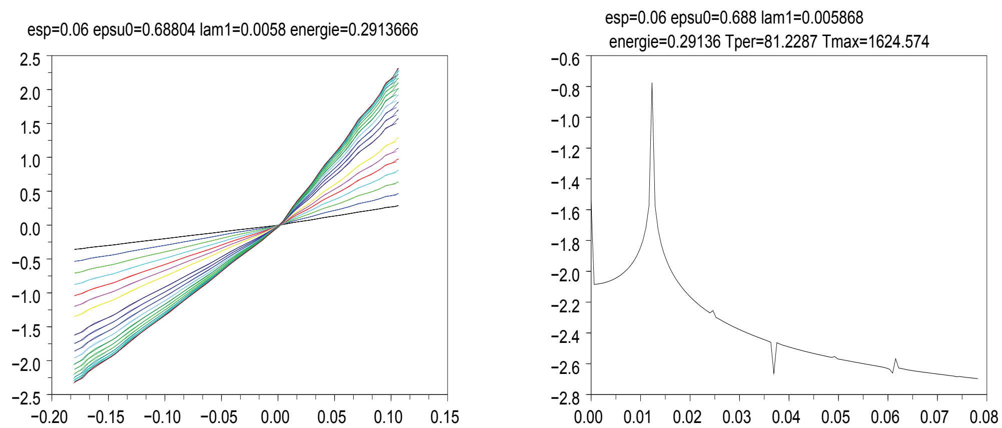

Figure 8: energy=0.29, 5 dof; left: configuration space; right:fft

4.4 First order asymptotic expansion

In this subsection, we do not particularize the initial data on one eigenmode. We adapt the method of strained coordinates since all modes are excited. We loose one order of accuracy compared to previous results since each mode does not stay periodic and becomes almost-periodic.
More precisely, the method of strained coordinates is used for each normal component, with the following initial data

$$u_k^\varepsilon(0) = a_k, \quad \dot{u}_k^\varepsilon(0) = 0, \qquad k = 1, \cdots, N.$$

Let us define N new times $s_k = \lambda_k^\varepsilon t$ and the following ansatz,

$$\begin{array}{rclcrcl} \lambda_k^\varepsilon & = & \lambda_k^0 + \varepsilon \lambda_k^1, & & \lambda_k^0 & = & \lambda_k, \\ u_k^\varepsilon(t) & = & v_k^\varepsilon(\lambda_k^\varepsilon t) & = & v_k^\varepsilon(s_k) & = & v_k^0(s_k) + \varepsilon r_k^\varepsilon(s_k). \end{array}$$

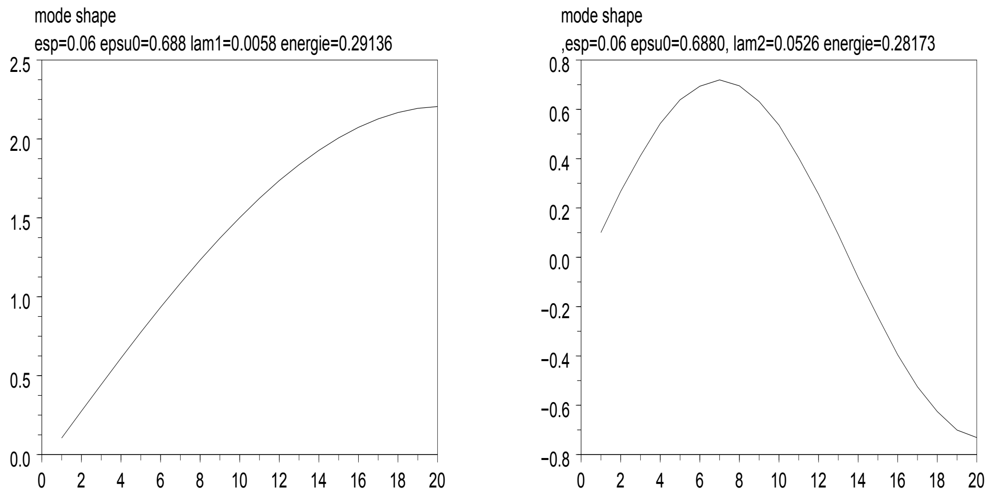

Figure 9: 20 dof; left:energy 0.29 mode 1; right: energy 0.28, mode 2

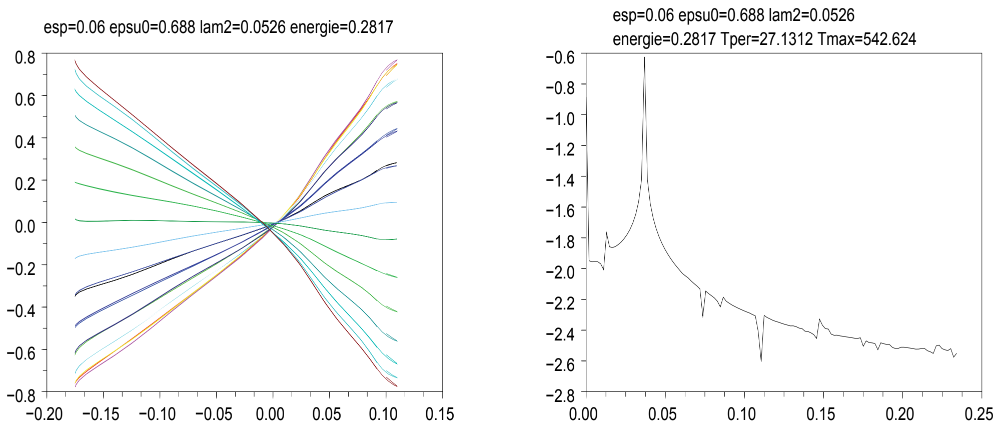

Figure 10: energy=0.28, 5 dof; left: configuration space; right:fft

The function v_k^0 are easily obtained by the linearized equation. Indeed, the only measured nonlinear effect for large time is given by $(\lambda_k^1)_{k=1}^N$. To obtain these N unknowns, we replace the previous ansatz in the system (25),

$$(\lambda_k^\varepsilon)^2(v_k^\varepsilon)''(s_k) + \lambda_k^2 v_k(s_k) \quad = \quad -\varepsilon\left(\sum_{j=1}^N a_{kj} v_j^\varepsilon\left(\frac{\lambda_j^\varepsilon}{\lambda_k^\varepsilon}s_k\right) - b_k\right)_+ .$$

The right hand side is written in variable s_k instead of s_j. Performing the expansion with respect to epsilon powers yields

$$L_k v_k^0 \quad = \quad (\lambda_k^0)^2(v_k^0)''(s_k) + \lambda_k^2 v_k^0(s_k) = 0, \tag{63}$$

$$-L_k r_k^\varepsilon(s_k) \quad = \quad \left(\sum_{j=1}^N a_{kj} v_j^0\left(\frac{\lambda_j^0}{\lambda_k^0}s_k\right) - b_k\right)_+ + 2\lambda_k\lambda_k^1(v_k^0)'' + R_k^\varepsilon. \tag{64}$$

Noting that replacing $v_j^\varepsilon(s_j)$ by $v_j^0\left(\frac{\lambda_j^0}{\lambda_k^0}s_k\right)$ in (64) implies a secular term of the order εt, since $s_j = \dfrac{\lambda_j^0}{\lambda_k^0}s_k + \mathcal{O}(\varepsilon t)$, the functions v_j^0 are smooth and the map $S \to S_+$ is one-Lipschitz. These new kind of errors $\mathcal{O}(\varepsilon t)$ are contained in the remainder of each right hand side:

$$R_k^\varepsilon(t) \quad = \quad \mathcal{O}(\varepsilon t) + \mathcal{O}(\varepsilon|r^\varepsilon|), \qquad |r^\varepsilon| = \sqrt{\sum_{k=1}^N (r_k^\varepsilon)^2}. \tag{65}$$

If $b_k = 0$, we identify the secular term with the Lemma 6.5 below and the relation $S_+ = S/2 + |S|/2$. Then, we remove the resonant term in the source term for the remainder r_k^ε, which gives us $\lambda_k^1 = \dfrac{a_{kk}}{4\lambda_k}$.
If $b_k \neq 0$, we compute λ_k^1 numerically with the following orthogonality condition to $\cos(s)$ written in the framework of almost periodic functions,

$$0 \quad = \quad \lim_{T\to\infty}\frac{1}{T}\int_0^T \left[\left(\sum_{j=1}^N a_{kj}v_j^0\left(\frac{\lambda_j^0}{\lambda_k^0}s_k\right) - b_k\right)_+ + 2\lambda_k\lambda_k^1 (v_k^0)''\right]\cdot\cos(s)ds.$$

The accuracy of the asymptotic expansion depends on the behavior of the solution $\phi = (\phi_1, \cdots, \phi_N)$ of the N following decoupled linear equations with right coefficients λ_k^1 to avoid resonance

$$-L_k\phi_k(s_k) \quad = \quad \left(\sum_{j=1}^N a_{kj}v_j^0\left(\frac{\lambda_j^0}{\lambda_k^0}s_k\right) - b_k\right)_+ + 2\lambda_k\lambda_k^1(v_k^0)''. \tag{66}$$

Furthermore each function r_k^ε depends on all times s_j, $j = 1, \cdots, N$ and becomes almost-periodic, i.e. $r_k^\varepsilon = r_k^\varepsilon(s_1, \cdots, s_N)$. Thus the method of strained coordinates, only working for periodic functions, fails to be continued.
Nevertheless, we obtain the following result proved in the Appendix.

Theorem 4.3 (All modes)
If $\lambda_1, \cdots, \lambda_N$ are $\mathbb{Z}$ independent, then, for any $T_\varepsilon = o(\varepsilon^{-1})$, i.e. such that

$$\lim_{\varepsilon\to 0} T_\varepsilon = +\infty, \quad \textit{and} \quad \lim_{\varepsilon\to 0}\varepsilon \times T_\varepsilon = 0,$$

we have for all $k = 1, \cdots, N$,

$$\lim_{\varepsilon\to 0}\|u_k^\varepsilon(t) - v_k^0\left(\lambda_k^\varepsilon\, t\right)\|_{W^{2,\infty}(0,T_\varepsilon)} = 0$$

where $\lambda_k^\varepsilon = \lambda_k + \varepsilon\lambda_k^1$, $v_k^0(s) = a_k\cos(s)$, and λ_k^1 is defined by:

$$\lambda_k^1 \quad = \quad \frac{1}{2\lambda_k a_0}\lim_{T\to+\infty}\frac{1}{T}\int_0^T \left(\sum_{j=1}^N a_{kj}v_j^0\left(\frac{\lambda_j^0}{\lambda_k^0}s_k\right) - b_k\right)_+ \cos(s)ds.$$

Furthermore, if $b_k = 0$, the previous integral yields: $\lambda_k^1 = \dfrac{a_{kk}}{4\lambda_k}$.

Notice that accuracy and large time are weaker than these obtained in Theorem 4.1. It is due to the inevitable accumulation of the spectrum near the resonance and the various times using in the expansion. On the other side we have the following direct improvement from the Theorem 4.1:

Remark 4.2 (Polarisation) *If only one mode are excited, for instance the number 1, i.e. $a_1 \neq 0$, $a_k = 0$ for all $k \neq 1$, then we have the estimate for all $t \in [0, \varepsilon^{-1}]$:*

$$\begin{array}{rcll} u_1^\varepsilon(t) = & v_1^0\left(\lambda_1^\varepsilon\, t\right) & +\mathcal{O}(\varepsilon). & \\ u_k^\varepsilon(t) = & 0 & +\mathcal{O}(\varepsilon) & \textit{for all } k \neq 1. \end{array}$$

5 Expansions with even periodic functions

Fourier expansion involving only cosines are used throughout this paper. There is never sinus. In this short section we explain why it is simple to work with even periodic functions and we give some hints to work with more general initial data.

First, we want to work only with co-sinus to avoid two secular terms. If we return to equation (12): $-\alpha_0(v_1''+v_1)=(v_0)_+ + \alpha_1 v_0''$. A priory, we have two secular terms in the right hand side, one with $\cos(s)$ and another with $\sin(s)$. Only one parameter α_1 seems not enough to delete all secular terms.
Otherwise, if $v_0\in\mathbb{R}$, u, S are 2π periodic even functions, $g\in C^0(\mathbb{R},\mathbb{R})$ such that $0=\displaystyle\int_0^{2\pi} e^{is}(S(s)+g(u(s)))ds$ then the solution of

$$v''+v \quad = \quad S(s)+g(u), \quad v(0)=v_0,\, v'(0)=0,$$

is necessarily a 2π periodic even function. Since we only work with 2π periodic even functions we have always at most one secular term proportional to $\cos(s)$.

We now investigate the case involving not necessarily even periodic functions. In general, $\dot{u}_0^\varepsilon\neq 0$ and u_ε is the solution of

$$\ddot{u}_\varepsilon+u_\varepsilon+\varepsilon f(u_\varepsilon)=0, \quad u_\varepsilon(0)=u_0^\varepsilon,\, \dot{u}_\varepsilon(0)=\dot{u}_0^\varepsilon.$$

By the energy $2E=\dot{u}^2+u^2+\varepsilon F(u)$, where $F'=2f$ and $F(0)=0$, we know that u_ε is periodic for ε small enough, for instance with an implicit function theorem see [28] also valid for Lipschitz function [4] in our case. Denote by τ_ε the first time such that $\dot{u}_\varepsilon(t)=0$. Such time exists thanks to the periodicity of u_ε. Now, let U_ε defined by $U_\varepsilon(t)=u_\varepsilon(t+\tau_\varepsilon)$. U_ε is the solution of

$$\ddot{U}_\varepsilon+U_\varepsilon+\varepsilon f(U_\varepsilon)=0, \quad U_\varepsilon(0)=U_0^\varepsilon=u_\varepsilon(\tau_\varepsilon),\, \dot{U}_\varepsilon(0)=0.$$

The initial data U_0^ε depends on the initial position and initial velocity of u_ε through the energy, $(U_0^\varepsilon)^2+\varepsilon F(U_0^\varepsilon)=(u_0^\varepsilon)^2+(\dot{u}_0^\varepsilon)^2+\varepsilon F(u_0^\varepsilon)$. For instance, if u_0^ε and $\dot{u}_\varepsilon^0$ are positive then U_0^ε is positive and
$U_0^\varepsilon=\sqrt{(u_0^\varepsilon)^2+(\dot{u}_0^\varepsilon)^2+\varepsilon(F(u_0^\varepsilon)-F\left(\sqrt{(u_0^\varepsilon)^2+(\dot{u}_0^\varepsilon)^2}\right)+\mathcal{O}(\varepsilon^2)}$.
We can apply the method of strained coordinates for U_ε only with even periodic functions: $U_\varepsilon(t)=v_0(\omega_\varepsilon t)+\varepsilon v_1(\omega_\varepsilon t)+\mathcal{O}(\varepsilon^2)$. The expansion obtained for u_ε by U_ε, with $\phi_\varepsilon=-\omega_\varepsilon\tau_\varepsilon$ is:

$$u_\varepsilon(t) \quad = \quad v_0(\omega_\varepsilon t+\phi_\varepsilon)+\varepsilon v_1(\omega_\varepsilon t+\phi_\varepsilon)+\mathcal{O}(\varepsilon^2),$$

which is a good ansatz in general for u_ε, where v_0 and v_1 are even $2\pi-$periodic functions. The method of strained coordinates becomes to find the following unknowns ϕ_0, ω_1,ϕ_1, ω_2,ϕ_2 such that

$$\begin{aligned}\omega_\varepsilon &= \omega_0+\varepsilon\omega_1+\varepsilon^2\omega_2+\cdots,\\ \phi_\varepsilon &= \phi_0+\varepsilon\phi_1+\varepsilon^2\phi_2+\cdots.\end{aligned}$$

Indeed, we have two parameters to delete two secular terms at each step. If one is only interested by the nonlinear frequency shift, it is simpler to work only with cosines.

Otherwise, if f is an odd function, we can work only with odd periodic function. It is often the case in literature when occurs a cubic non-linearity. See for instance [16, 17, 18] for the Duffing equation, the Rayleigh equation or the Korteweg-de Vries equation.

6 Appendix: technical proofs

We give some useful results about energy estimates and almost periodic functions in subsection 6.1. Next we complete the proofs for each previous asymptotic expansions in subsection 6.2. The point is to bound the remainder for large time in each expansion.

6.1 Useful lemmas

The following Lemma is useful to prove an expansion for large time with non smooth nonlinearity.

Lemma 6.3 [Bounds for large time]
Let w_ε be a solution of

$$\left\{ \begin{array}{l} w_\varepsilon'' + w_\varepsilon = S(s) + f_\varepsilon(s) + \varepsilon g_\varepsilon(s, w_\varepsilon), \\ w_\varepsilon(0) = 0, \quad w_\varepsilon'(0) = 0. \end{array} \right. \tag{67}$$

If source terms satisfy the following conditions where $M > 0$, $C > 0$ are fixed constants :

1. *$S(s)$ is a 2π-periodic function orthogonal to $e^{\pm is}$, and $|S(s)| \leq M$ for all s,*
2. *$|f_\varepsilon| \leq M$ and for all T, $\displaystyle\int_0^T |f_\varepsilon(s)|ds \leq C\varepsilon T$ (resp. $C\sqrt{\varepsilon}T$),*
3. *for all $R > 0$: $M_R = \displaystyle\sup_{\varepsilon\in(0,1), s>0, R>|u|} |g_\varepsilon(s, u)| < \infty$,*
   that is to say that $g_\varepsilon(s, u)$ is locally bounded with respect to u independently from $\varepsilon \in (0, 1)$ and $s \in (0, +\infty)$,

then, there exists $\varepsilon_0 > 0$ and $\gamma > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, w_ε is uniformly bounded in $W^{2,\infty}(0, T_\varepsilon)$, where $T_\varepsilon = \dfrac{\gamma}{\varepsilon}$ (resp. $\dfrac{\gamma}{\sqrt{\varepsilon}}$).

Notice that f_ε and g_ε are not necessarily continuous. Indeed this a case for our asymptotic expansion, see Lemme 3.1 and its applications throughout the paper. But in previous sections the right hand side is globally continuous, i.e. $S + f_\varepsilon + \varepsilon g_\varepsilon(., w_\varepsilon)$ is continuous, so, in this case, w_ε is C^2.

Proof of the Lemma 6.3: First we remove the non resonant periodic source term which is independent of ε. Second, we get L^∞ bound for w_ε and w_ε' with an energy estimate. Third, with equation (67), we get an uniform estimate for w_ε'' in $L^\infty(0, T_\varepsilon)$ and the $W^{2,\infty}$ regularity.
Step 1: remove S
It suffices to write $w_\varepsilon = w_1 + w_2^\varepsilon$ where w_1 solves the linear problem:

$$w_1'' + w_1 = S(s), \quad w_1(0) = 0, w_1'(0) = 0. \tag{68}$$

w_1 and w_1' are uniformly bounded in $L^\infty(0, +\infty)$ since there is no resonance.
More precisely, $w_1 = F(s) + A\cos(s) + B\cos(s)$, where F is 2π periodic. F is obtained by Fourier expansion without harmonic $n = \pm 1$ since S is never resonant:

$$F(s) \;=\; \sum_{n\neq\pm1} \frac{c_n}{1-n^2} e^{ins} \quad \textit{with} \quad S(s) = \sum_{n\neq\pm1} c_n e^{ins}.$$

F is uniformly bounded, with Cauchy-Schwartz inequality set $C_0^2 = \displaystyle\sum_{n\neq\pm1} |n^2 - 1|^{-2}$, we obtain:

$$\|F\|_{L^\infty} \;\leq\; \sum_{n\neq\pm1} \frac{|c_n|}{|n^2-1|} \leq C_0 \|S\|_{L^2(0,2\pi)} \leq C_0 \|S\|_{L^\infty(0,2\pi)}.$$

Similarly, set $D_0^2 = \sum_{n \neq \pm 1} n^2 |n^2-1|^{-2}$, we have $\|F'\|_{L^\infty} \le D_0 \|S\|_{L^\infty(0,2\pi)}$.
Furthermore, $0 = w_1(0) = F(0) + A$, and $0 = (w_1)'(0) = F'(0) + B$, then, A and B are well defined. $w_1^{\cdot}$ is also bounded, i.e. there exists $M_1 > 0$ such that $\|w_1\|_{W^{1,\infty}(0,+\infty)} \le M_1$.
Notice that from equation (68), w_1 belongs in $W^{2,\infty}$.

Then we get an equation similar to (67) for w_2^ε with $S \equiv 0$ and the same assumption for the same f_ε and the new g_ε: $\overline{g}_\varepsilon(s,w) = g_\varepsilon(s, w_1 + w)$.

$$\left\{ \begin{array}{l} (w_2^\varepsilon)'' + (w_2^\varepsilon) = f_\varepsilon(s) + \varepsilon \overline{g}_\varepsilon(s, w_2^\varepsilon), \\ (w_2^\varepsilon)(0) = 0, \quad (w_2^\varepsilon)'(0) = 0. \end{array} \right. \tag{69}$$

Step 2: energy estimate
Second, we get an energy estimate for w_2^ε. We fix $R > 0$ such that R is greater than the uniform bound M_1 obtained for w_ε^1, $R = M_1 + \rho$ with $\rho > 0$. Let us define

$$2E(s) = ((w_2^\varepsilon)'(s))^2 + (w_2^\varepsilon)(s)^2, \qquad \overline{E}(s) = \sup_{0<\tau<s} E(\tau),$$

and T_ε be the first time $T > 0$ such that $2\overline{E}(T) \ge \rho^2$, i.e. ρ estimates the size of (w_2^ε) and $(w_2^\varepsilon)'$.
Multiplying the differential equation (69) by $(w_2^\varepsilon)'$, we have for all $s < T < T_\varepsilon(\rho)$ the following inequalities since $\sup_{0<\tau<s} |(w_2^\varepsilon)'(\tau)| \le \sqrt{2\overline{E}(s)}$, and $\int_0^T |f_\varepsilon(s)| ds \le C\varepsilon T$,

$$\begin{aligned} E(s) &= \int_0^s f_\varepsilon(\tau)(w_2^\varepsilon)'(\tau) d\tau + \varepsilon \int_0^s \overline{g}_\varepsilon(\tau, (w_2^\varepsilon)(\tau))(w_2^\varepsilon)'(\tau) d\tau, \\ &\le C\varepsilon s \sqrt{2\overline{E}(s)} + \varepsilon s M_R \sqrt{2\overline{E}(s)}, \\ \overline{E}(T) &\le C\varepsilon T \sqrt{2\overline{E}(T)} + \varepsilon T M_R \sqrt{2\overline{E}(T)}, \\ \varepsilon T &\ge \frac{\sqrt{\overline{E}(T)/2}}{M_R + C}. \end{aligned}$$

Notice that if $2\overline{E}(T) < \rho^2$ for all $T > 0$ then $T_\varepsilon = +\infty$. The critical case is when T_ε is finite and $\overline{E}(T)$ approaches $\rho^2/2$ when T goes to $T_\varepsilon(\rho)$. Thus we have $T_\varepsilon \ge \dfrac{\rho}{2\varepsilon(M_R + C)}$ and $E(t) \le \frac{\rho^2}{2}$ for $t \le T_\varepsilon = \frac{\gamma}{\varepsilon}$ with $\gamma = \frac{\rho}{2(M_R + C}$.
The proof is similar when $\int_0^s |f_\varepsilon(\tau)| d\tau \le C\sqrt{\varepsilon} T$ then $T_\varepsilon \ge \dfrac{\rho}{2\sqrt{\varepsilon}(\sqrt{\varepsilon} M_R + C)}$. □

For completeness, we state a similar and straightforward version of Lemma 6.3 useful for systems.

Lemma 6.4 [Bounds for large time for systems]
Let $w_\varepsilon = (w_1^\varepsilon, \cdots, w_N^\varepsilon)$ be the solution of the following system:

$$\left\{ \begin{array}{l} (\lambda_1)^2 (w_k^\varepsilon)'' + (\lambda_k)^2 w_k^\varepsilon = S_k(s) + f_k^\varepsilon(s) + \varepsilon g_k^\varepsilon(s; w_\varepsilon), \\ w_k^\varepsilon(0) = 0, \quad (w_k^\varepsilon)'(0) = 0, \quad k = 1, \cdots, N. \end{array} \right. \tag{70}$$

If source terms satisfy the following conditions where $M > 0$, $C > 0$ are fixed constants :

1. *non resonance conditions with $S_k(s)$ are 2π-periodic functions and $|S_k(s)| \le M$,*
   (a) *$S_1(s)$ is orthogonal to $e^{\pm is}$, i.e. $\int_0^{2\pi} S_1(s) e^{\pm is} ds = 0$,*
   (b) *λ_k, λ_1 are $\mathbb{Z}$ independent for all $k \neq 1$,*

2. $|f_k^\varepsilon| \le M$ *and for all* T*,* $\displaystyle\int_0^T |f_\varepsilon(s)|ds \le C\varepsilon T$ *or* $C\sqrt{\varepsilon}T$*,*

3. for all $R > 0$*:* $M_R = \max_k \sup_{\varepsilon\in(0,1),s>0,w_1^2+\cdots+w_N^2<R^2} |g_k^\varepsilon(s;u)| < \infty$*,*

then, there exists $\varepsilon_0 > 0$ *and* $\gamma > 0$ *such that, for* $0 < \varepsilon < \varepsilon_0$*,* w_ε *is uniformly bounded in* $W^{2,\infty}(0, T_\varepsilon)$*, where* $T_\varepsilon = \dfrac{\gamma}{\varepsilon}$ *or* $\dfrac{\gamma}{\sqrt{\varepsilon}}$*.*

Proof : First we remove source terms S_k independent of ε setting $w_k^\varepsilon = w_{k,1} + w_{k,2}^\varepsilon$ where $w_{k,1}$ is the solution of

$$\lambda_1^2 w_{k,1}'' + \lambda_k^2 w_{k,1} = S_k, \quad w_{k,1}(0) = 0, \quad w_{k,1}'(0) = 0.$$

As in the proof of Lemma 6.3, $w_{1,1}$ belongs in $W^{2,\infty}$ thanks to the non-resonance condition *1.(a)*. For $k \neq 1$, there is no resonance since $\frac{\lambda_k}{\lambda_1} \notin \mathbb{Z}$, i.e. the non-resonance condition *1.(b)*, thus a similar expansion also yields $w_{k,1}$ belongs in $W^{2,\infty}(\mathbb{R}, \mathbb{R})$.

Now $w_{k,2}^\varepsilon$ are solutions of the following system for $k = 1, \cdots, N$

$$\begin{cases} \lambda_1^2 (w_{k,2}^\varepsilon)'' + \lambda_k^2 (w_{k,2}^\varepsilon) = f_k^\varepsilon(s) + \varepsilon \overline{g}_k^\varepsilon(s; w_2^\varepsilon), \\ (w_{k,2}^\varepsilon)(0) = 0, \quad (w_{k,2}^\varepsilon)'(0) = 0, \end{cases}$$

with $w_\varepsilon = w_1 + w_2^\varepsilon$, $w_2^\varepsilon = (\cdots, w_{k,2}^\varepsilon, \cdots)$ and $\overline{g}_k^\varepsilon(s; \cdots, w_k, \cdots) = g_k^\varepsilon(s; \cdots, w_{k,1} + w_k, \cdots)$. The end of the proof of Lemma 6.4 is a straightforward generalization of the the proof of Lemma 6.3 with the energy: $2E(w_1, \cdots, w_N) = \displaystyle\sum_{k=1}^N \left((\lambda_1)^2 (\dot{w}_k)^2 + (\lambda_k)^2 w_k^2\right).$ □

For systems, we also have to work with linear combination of periodic functions with different periods and nonlinear function of such sum. So we work with the adherence in $L^\infty(\mathbb{R}, \mathbb{C})$ of span$\{e^{i\lambda t}, \lambda \in \mathbb{R}\}$, namely the set of almost periodic functions $C_{ap}^0(\mathbb{R}, \mathbb{C})$, and the Hilbert space of almost-periodic function is $L_{ap}^2(\mathbb{R}, \mathbb{C})$, see [6], with the scalar product

$$\langle u, v \rangle = \lim_{T\to+\infty} \frac{1}{T} \int_0^T u(t)\overline{v(t)}dt.$$

We give an useful Lemma about the spectrum of $|u|$ for $u \in C_{ap}^0(\mathbb{R}, \mathbb{R})$. Let us recall definitions for the Fourier coefficients of u associated to frequency λ: $c_\lambda[u]$ and its spectrum: $Sp\ [u]$,

$$c_\lambda[u] = \left\langle u, e^{i\lambda t}\right\rangle = \lim_{T\to+\infty} \frac{1}{T} \int_0^T u(t)e^{-i\lambda t}dt, \qquad Sp\ [u] = \{\lambda \in \mathbb{R}, c_\lambda[u] \neq 0\}. \quad (71)$$

Lemma 6.5 [Property of the spectrum of $|u|$]
If $u \in C_{ap}^0(\mathbb{R}, \mathbb{R})$*, and if* u *has got a finite spectrum:*
$Sp\ [u] \subset \{\pm\lambda_1, \cdots, \pm\lambda_N\}$*, with* $(\lambda_1, \cdots, \lambda_N)$ *being* $\mathbb{Z}$*-independent, such that* $0 \notin Sp\ [u]$*,*
then $\lambda_k \notin Sp\ [\ |u|\ ]$ *for all* k*.*

Proof : The result is quite obvious for u^2. We first prove the result for $f(u^2)$ where f is smooth. Then, we conclude by approximating $|u|$ by a smooth sequence $f_n(u^2) = \sqrt{1/n + u^2}$, and using the L^∞ stability of the spectrum.
Let E be the set of all $\mathbb{Z}$ linear combinations of elements of $S_2 = \{0, \pm\lambda_{kj}^\pm, k, j = 1, \cdots N\}$, where

$$\lambda_{kj}^\pm = \lambda_k \pm \lambda_j.$$

Thus $Sp\ [f(u^2)]$ is a subset of E since $Sp\ [u^2] \subset S_2$.
Notice that $\lambda_{jk}^\pm = \pm\lambda_{kj}^\pm, \quad \lambda_{kk}^- = 0, \quad \lambda_{kk}^+ = 2\lambda_k = \lambda_{kj}^+ + \lambda_{kj}^-.$

Choosing $k = 1$ for instance, so $\lambda_1 \neq 0$, it suffices to prove that $\lambda_1 \notin E$.
Assume the converse, i.e., $\lambda_1 \in E$. Then, for $k < j$, there exists some integers $(c_{kj}^{\pm})_{k<j)}$ such that:

$$\lambda_1 \quad = \quad \sum_{k<j} (c_{kj}^{+}\lambda_{kj}^{+} + c_{kj}^{-}\lambda_{kj}^{-}).$$

Therefore, defining $c_{jk}^{\pm}$ by $\pm c_{kj}^{\pm}$ for $k < j$, we have:

$$\lambda_1 \quad = \quad \lambda_1 \sum_{j\neq 1}(c_{1j}^{+} + c_{1j}^{-}) + \lambda_2 \sum_{j\neq 2}(c_{2j}^{+} + c_{2j}^{-}) + \cdots + \lambda_N \sum_{j\neq 1}(c_{k=Nj}^{+} + c_{Nj}^{-}).$$

Using the $\mathbb{Z}$-independence, with $d_{kj} = c_{kj}^{+} + c_{kj}^{-}$ for $k \neq j$ and $d_{kk} = 0$, we have following system: $1 = D_1 = \sum_j d_{1j}, \quad 0 = D_k = \sum_j d_{kj}, \qquad$ for all $k > 1$.

Summing up, the $N-1$ last equations in $\dfrac{\mathbb{Z}}{2\mathbb{Z}}$, and using the fact: $d_{jk} \equiv d_{kj}$ modulo 2, we have: $0 \equiv \sum_{k=2}^{N} D_k \equiv \sum_{j=2}^{N} d_{1j} + 2\sum_{k<j} d_{kj} \equiv \sum_{j=2}^{N} d_{1j}$, then $D_1 \equiv 0$, i.e. D_1 is even. It's impossible since $D_1 = 1$. So $\lambda_1 \notin E$ and the proof is complete. $\square$

6.2 Bounds for the remainders

Now, we prove each asymptotic expansion given in previous sections, i.e. we bound each remainders with energy estimates up to a large time.

Proof of Proposition 2.1 : First we give the outline of the proof.
Notice that all these computations only involve the function cos. Then, the only way to have a secular term in equations defining v_1 and v_2 is a $\cos(s)$ in the right-hand side. So, the good choice of α_1 and α_2, is enough to remove secular term with $\cos(s)$. Now, it suffices to control r_ε for large time. A computation shows that the remainder R_ε of equation (13) satisfies:
$|R_\varepsilon(s)| \leq C\varepsilon(1 + |r_\varepsilon(s)|) + |\chi_\varepsilon|(v_0, v_1 + \varepsilon r_\varepsilon)$.
Then, r_ε is like w_ε in Lemma 6.3, and the term f_ε comes from χ_ε which is estimated by Lemmas 3.1, 3.2.
More precisely, an exact computation of R_ε in equation (13) leads to

$$R_\varepsilon \quad = \quad \chi_\varepsilon(v_0, v_1 + \varepsilon r_\varepsilon) + \varepsilon H(v_0) r_\varepsilon + \varepsilon \alpha_3^\varepsilon v_\varepsilon'',$$

where α_3^ε is a real constant, bounded uniformly for all $\varepsilon \in [0, 1]$ such that
$(\omega_\varepsilon)^2 = \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \varepsilon^3\alpha_3^\varepsilon$. From (23) we also have

$$\begin{aligned} & \chi_\varepsilon(v_0, v_1 + \varepsilon r_\varepsilon) \\ = \quad & \frac{(v_0 + \varepsilon v_1 + \varepsilon^2 r_\varepsilon)_+ - [(v_0)_+ + H(v_0)(\varepsilon v_1 + \varepsilon^2 r_\varepsilon)]}{\varepsilon} \\ = \quad & \tfrac{(v_0+\varepsilon v_1)_+ - [(v_0)_+ + \varepsilon H(v_0) v_1]}{\varepsilon} + \tfrac{(v_0+\varepsilon v_1+\varepsilon^2 r_\varepsilon)_+ - (v_0+\varepsilon v_1)_+}{\varepsilon} - \varepsilon H(v_0) r_\varepsilon \\ = \quad & \chi_\varepsilon(v_0, v_1) - \varepsilon H(v_0) r_\varepsilon + \varepsilon \tilde{g}_\varepsilon(s, r_\varepsilon), \end{aligned}$$

since $u \to (u)_+$ is 1-Lipschitz

$$\begin{aligned} \tilde{g}_\varepsilon(s, r_\varepsilon) \quad &= \quad \frac{(v_0 + \varepsilon v_1 + \varepsilon^2 r_\varepsilon)_+ - (v_0 + \varepsilon v_1)_+}{\varepsilon}, \\ |\tilde{g}_\varepsilon(s, r_\varepsilon)| \quad &\leq \quad |r_\varepsilon|. \end{aligned}$$

So, with $v_\varepsilon = v_0 + \varepsilon v_1 + \varepsilon^2 r_\varepsilon$, we can rewrite R_ε as follow

$$R_\varepsilon \quad = \quad \chi_\varepsilon(v_0, v_1) + \varepsilon \tilde{g}_\varepsilon(s, r_\varepsilon) + \varepsilon \alpha_3^\varepsilon v_\varepsilon''.$$

Now, we can rewrite equation (13) in the following way

$$\begin{aligned}
-\alpha_0(r_\varepsilon'' + r_\varepsilon) &= S(s) + f_\varepsilon(s) + \varepsilon g_\varepsilon(s, r_\varepsilon), \\
S &= \alpha_2 v_0'' + \alpha_1 v_1'' + H(v_0)v_1, \\
f_\varepsilon &= \chi_\varepsilon(v_0, v_1) + \varepsilon\alpha_3^\varepsilon(v_0" + \varepsilon v_1"), \\
g_\varepsilon &= \tilde{g}_\varepsilon + \varepsilon^2\alpha_3^\varepsilon r_\varepsilon,
\end{aligned}$$

which allow us to conclude with Lemma 6.3. □

The proof for other propositions 2.2, 2.3 in section 2 are similar.
We now complete the proof for the asymptotic expansions for systems given in section 4.

Proof of Theorem 4.1 : As in the proof of Proposition 2.1, the same technique of proof is used component by component for Theorems 4.1, with similar energy estimates we can conclude with the Lemma 6.4 for system to control all r_k^ε.
More precisely a complete computation of the remainder gives us:

$$R_k^\varepsilon = \chi_\varepsilon(a_{k1}v_1^0 - b_k, \sum_j a_{kj}[v_{kj}^1 + \varepsilon r_j^\varepsilon]) + \varepsilon H(a_{k1}v_1^0 - b_k)\sum_j a_{kj} r_j^\varepsilon + \varepsilon\alpha_3^\varepsilon(v_k^\varepsilon)'',$$

with notation of the proof of Proposition 2.1 and $v_k^\varepsilon = v_k^0 + \varepsilon v_k^1 + \varepsilon^2 r_k^\varepsilon$. Let u, v, w be three functions, as previously, we have:

$$\chi_\varepsilon(u, v + \varepsilon w) + \varepsilon H(u)w = \chi_\varepsilon(u, v) + \varepsilon^{-1}((u + \varepsilon v + \varepsilon^2 w)_+ - (u + \varepsilon v)_+),$$

and, since $w \to w_+$ is 1-Lipschitz:

$$|\varepsilon^{-1}((u + \varepsilon v + \varepsilon^2 w)_+ - (u + \varepsilon v)_+)| \leq \varepsilon|w|.$$

Now, we can rewrite R_k^ε as follow:

$$R_k^\varepsilon = \chi_\varepsilon(a_{k1}v_1^0 - b_k, \sum_j a_{kj}v_{kj}^1) + \varepsilon g_k^\varepsilon(s, r_1^\varepsilon, \cdots, r_N^\varepsilon) + \varepsilon\alpha_3^\varepsilon(v_k^\varepsilon)'',$$

where g_k^ε is defined by

$$\begin{aligned}
g_k^\varepsilon(s, r_1^\varepsilon, \cdots, r_N^\varepsilon) &= \varepsilon^{-1}\{(V_k^\varepsilon + \varepsilon^2\sum_j a_{kj}r_j^\varepsilon)_+ - (V_k^\varepsilon)_+\}, \\
V_k^\varepsilon &= a_{k1}v_k^0 - b_k + \varepsilon\sum_j a_{kj}v_{kj}^1,
\end{aligned}$$

Notice that g_k^ε satisfies $|g_k^\varepsilon(s, r_1^\varepsilon, \cdots, r_N^\varepsilon)| \leq \sum_j |a_{kj}||r_j^\varepsilon|$.
A key ingredient is the energy $2E = \sum_k(\alpha_0(r_k')^2 + \lambda_k^2 r_k^2)$ for the homogeneous system: $L_k r_k = 0,\ k = 1, \cdots, N$ and the for the inhomogeneous system:

$$-L_k r_k = S_k^\varepsilon(s) + f_k^\varepsilon(s) + \varepsilon g_k(s, r_1^\varepsilon, \cdots, r_N^\varepsilon),$$

for $k = 1, \cdots, N$, with

$$S_k = H(a_{k1}v_1^0 - b_k)\sum_{j=1}^N a_{kj}v_j^1 + \alpha_2 v_k^{0"} + \alpha_1 v_k^{1"},$$

and α_1, α_2 are well chosen to avoid secular term when $k = 1$. Thus, all S_k are 2π periodic. S_1 is not resonant with L_1. The λ_k are $\mathbb{Z}$ independent. We can apply Lemma 6.4 which is enough to conclude the proof. □

Proof of Theorem 4.3 : The proof follows two steps. First the solution for linear equations (66) are bounded by $o(t)$. Second, energy estimates are used to bound r^ε.
At the end we prove remark 4.2.
Notice that we do not use Lemmas 3.1, 3.2. Indeed, we have no term with χ_ε. We only use that functions u_+ and v_k^0 are Lipschitz, the Lemma 6.5 to identify resonant terms when $b_k = 0$ and an energy estimate. But, since all modes are excited, the accuracy is weaker than the precision obtained in Theorem 4.1, as in [24].

Step 1: the N problems (66) involves decoupled equations rewritten as follow with $\omega > 0$,

$$\phi''(s) + \omega^2\phi(s) \quad = \quad S(s) \in C^0_{ap}(\mathbb{R}, \mathbb{R}), \qquad \pm\omega \notin Sp[S].$$

There is no resonance since $\pm\omega$ are not in the spectrum of S. But, $Sp[S]$ is dense in $\mathbb{R}$. Indeed $\lambda_1, \cdots, \lambda_N$ are $\mathbb{Z}$ independent. In general, we cannot expect that ϕ is bounded on the real line, see [6], but ϕ is less than $\mathcal{O}(s)$ for large time. We can compute explicitly ϕ

$$\begin{aligned} \phi(s) &= A\cos(\omega s) + B\sin(\omega s) + \psi(s), \\ \omega\psi(s) &= \int_0^s S(\sigma)\sin(\omega(s-\sigma))d\sigma \\ &= \sin(\omega s)\int_0^s S(\sigma)\cos(\omega\sigma)d\sigma - \cos(\omega s)\int_0^s S(\sigma)\sin(\omega\sigma)d\sigma. \end{aligned}$$

The condition $\pm\omega \notin Sp[S]$ is $\displaystyle\lim_{s\to+\infty} s^{-1}\int_0^s S(\sigma)\exp(\pm i\omega\sigma)d\sigma = 0$. That is to say $\displaystyle\int_0^s S(\sigma)\exp(\pm i\omega\sigma)d\sigma = \ o(s)$ when $s \to +\infty$, thus ψ and ϕ are negligible compared to s for large time.

Step 2: Let us decompose the remainder in the following way $r_k^\varepsilon = \phi_k + w_k^\varepsilon$. From equation (65) and the previous bound for ϕ_k we have in variable t instead of s_k for convenience

$$L_k w_k^\varepsilon(t) \quad = \quad \mathcal{O}(\varepsilon t) + (\mathcal{O}(\varepsilon\phi_k) + \mathcal{O}(\varepsilon|w^\varepsilon|)) = \mathcal{O}(\varepsilon t) + \mathcal{O}(\varepsilon|w^\varepsilon|),)$$

since $\phi_k(t) = o(t)$. Now, we remove the first part of the right hand side with $w_k^\varepsilon = \tilde{w}_k^\varepsilon + z_k^\varepsilon$ and $\tilde{w}_k^\varepsilon$ is solution of $L_k\tilde{w}_k^\varepsilon = \mathcal{O}(\varepsilon t)$. Classical energy estimates (or explicit computations as for ϕ) yields to $\tilde{w}_k^\varepsilon(t) = \mathcal{O}(\varepsilon t^2)$. Thus there exists a constant $C_1 > 0$ such that z_k^ε satisfies

$$|L_k z_k^\varepsilon| \quad \leq \quad C(\varepsilon^2 t^2 + \varepsilon|z^\varepsilon|).$$

Multiplying each inequality by $|(z_k^\varepsilon)'|$, summing up with respect to k, integrating on $[0, T]$, by Cauchy-Schwarz inequality, with $D = 2C(\min(\lambda_k) + \min(\lambda_k)^2)$ we get

$$\begin{aligned} E(T) &= \sum_{k=1}^N \left(\lambda_1^2((z_k^\varepsilon)')^2 + \lambda_k^2(z_k^\varepsilon)^2\right) \\ &\leq 2C\varepsilon^2T^2\int_0^T \sum_{k=1}^N |(z_k^\varepsilon)'|(t)dt + 2C\varepsilon\int_0^T |(z^\varepsilon)'\cdot z^\varepsilon|dt \\ &\leq D\varepsilon^2T^{2.5}\sqrt{\int_0^T E(t)dt} + D\varepsilon\int_0^T E(t)dt. \end{aligned}$$

Let $Y(T)$ be $\displaystyle\int_0^T E(t)dt$, thus $Y(0) = 0$ and for all $t \in [O, T]$,

$$E(t) = Y'(t) \quad \leq \quad D\varepsilon^2T^{2.5}\sqrt{Y(t)} + D\varepsilon Y(t).$$

Since $\int_0^Y \frac{dy}{A\sqrt{y}+y} = 2\ln\left(1+\frac{\sqrt{y}}{A}\right)$ we obtain $\sqrt{Y(T)} \leq \varepsilon T^{2.5}\exp(D\varepsilon T)$ and then

$$E(T) \quad \leq \quad 2D\varepsilon^3 T^5 \exp(D\varepsilon T).$$

Finally $r_k^\varepsilon = \phi_k + \tilde{w}_k^\varepsilon + z_k^\varepsilon = o(T) + \mathcal{O}(\varepsilon T^2) + \mathcal{O}(\varepsilon^{1.5}T^{2.5}\exp(D\varepsilon T))$, so for any $T_\varepsilon = o(\varepsilon^{-1})$ we have in $W^{1,\infty}(0,T_\varepsilon)$ for all $T \leq T_\varepsilon$

$$\varepsilon r_\varepsilon(T) \quad = \quad o(\varepsilon T_\varepsilon) + \mathcal{O}(\varepsilon^2 T_\varepsilon^2) + \mathcal{O}(\varepsilon^{2.5}T_\varepsilon^{2.5}),$$

which is enough to have the convergence in $W^{1,\infty}(0,T_\varepsilon)$. Furthermore r_k^ε satisfies the second order differential equation (64) which is enough to get the convergence in $W^{2,\infty}$.

About remark 4.2: From Theorem 4.3, this result its obvious. Let us explain why we cannot go further up to the order ε^2.

Unfortunately S_k is not periodic since v_j^1 is quasi-periodic for $j \neq 1$. Indeed, the following initial conditions

$$v_k^1(0) = 0, \quad (v_k^1)'(0) = 0, \quad k \neq 1,$$

yields to a quasi-periodic function, sum of two periodic functions with different periods 2π and $2\pi\lambda_1/\lambda_k$, thus a globally bounded function

$$v_k^1(s) \quad = \quad \phi_k^1(s) - \phi_k^1(0)\cos\left(\frac{\lambda_k}{\lambda_1}s\right).$$

So we cannot apply Lemma 6.4.
Let us decompose $S_k = P_k + Q_k$ for $k \neq 2$ where P_k is periodic and Q_k is almost-periodic

$$Q_k(s) \quad = \quad -H(a_{k1}v_1^0(s) - b_k)\sum_{j=1}^N a_{kj}\phi_j^1(0)\cos\left(\frac{\lambda_j}{\lambda_1}s\right).$$

Let w_k be a solution of $-L_k w_k = Q_k$ then $Sp[w_k] \in \bigcup_j \left\{\pm\frac{\lambda_j}{\lambda_1} + \mathbb{Z}\right\}$, so the spectrum of w_k is discrete and there is resonance in the $N-1$ equations, $-L_k r_k^\varepsilon = S_k + \cdots$, $k \neq 1$ and the expansion does not still valid for time of the order ε^{-1}. $\square$

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