Regularity of optimal transport on compact, locally nearly spherical, manifolds

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Philippe Delanoë and Yuxin Ge†

Abstract

Given a couple of smooth positive measures of same total mass on a compact connected Riemannian manifold M, we look for a smooth optimal transportation map G, pushing one measure to the other at a least total squared distance cost, directly by using the continuity method to produce a classical solution of the elliptic equation of Monge–Ampère type satisfied by the potential function u, such that $G = \exp(\operatorname{grad} u)$. This approach boils down to proving an *a priori* upper bound on the Hessian of u, which was done on the flat torus by the first author. The recent local C^2 estimate of Ma–Trudinger–Wang enabled Loeper to treat the standard sphere case by overcoming two difficulties, namely: in collaboration with the first author, he kept the image $G(m)$ of a generic point $m \in M$, uniformly away from the cut-locus of m; he checked a fourth-order inequality satisfied by the squared distance cost function, proving the uniform positivity of the so-called c-curvature of M. In the present paper, we treat along the same lines the case of manifolds with curvature sufficiently close to 1 in C^2 norm – specifying and proving a conjecture stated by Trudinger.

Introduction

We are interested in the regularity of the optimal transportation map G which pushes a given positive Borel measure $\mu_0 = \rho_0 \mathrm{dVol}$ to another one $\mu_1 = \rho_1 \mathrm{dVol}$ of same total mass on a compact connected n-dimensional Riemannian manifold (M_n, g) with Lebesgue measure dVol, when all data are smooth and the cost-function c is the Brenier–McCann one [5, 6, 34], namely:

$$\forall (p,q) \in M_n^2,\ c(p,q) = \frac{1}{2} d_g^2(p,q),$$

d_g standing for the geodesic distance in (M_n, g). The map G minimizes the total cost functional

$$\mathcal{C}(\Phi) = \int_{M_n} c[p, \Phi(p)]\ d\mu_0$$

among measurable maps $\Phi : M_n \to M_n$ which push μ_0 to μ_1 (meaning $\mu_1(B) = \mu_0[\Phi^{-1}(B)]$ for each Borel subset $B \subset M_n$, written $\Phi_\# \mu_0 = \mu_1$). The existence

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of a unique such minimizing map G is established in the landmark paper [34]. The smoothness of G is known in the following cases:

(i) anytime the densities ρ_0, ρ_1 are close enough in $C^\infty(M_n)$ [16, p.157]; in $C^{0,\alpha}(M_n)$ for some $\alpha \in (0,1)$ is enough to have $G \in C^{1,\alpha}$;

(ii) given measures μ_0, μ_1 as above, anytime the metric g is C^∞-close enough to a metric for which the optimal map is smooth [16, p.159]; $C^{2,\alpha}$-close would suffice to get $G \in C^{1,\alpha}$;

(iii) if the metric g is flat [16] (see also [7, 8, 9, 14]);

(iv) on the standard sphere [31];

(v) if the c-curvature is positive (a 4th-order condition on the cost-function c put forward in [33], also expressed in Equations (2)-(4) below) and if the exponential map is non-singular on the tangent cut-locus [32].

Here, let us observe that the result (iv) implies, by naturality and uniqueness, that the optimal transportation map G is also smooth on any manifold (M_n, g) with constant positive curvature; this was independently observed by Young–Heon Kim. Further regularity results in that spirit are announced in [28] (see also Appendix C below). Besides, let us note that the second condition of the regularity result (v) precludes positively curved simply connected manifolds (with $\frac{1}{4}$-pinching if odd-dimensional) [29, 30] (see also [40, 1]).
In contrast with the preceding results, if the curvature of g is *not* non-negative on M_n, one cannot expect G smooth for *arbitrary* smooth positive measures μ_0, μ_1 [31]. Worse, it was recently shown that positive curvature alone does *not* imply G smooth [27, 32].
Neil Trudinger has conjectured that the smoothness of G should be derivable from the positivity of the curvature provided the k-th covariant derivatives of the curvature tensor are assumed to be small enough for $1 \leq k \leq r$ with a suitable integer $r \geq 2$ (which trivially holds on the standard sphere). Considering the results (iv) and (ii) above, the issue here is to quantify how far the curvature tensor may differ from a spherical one and to show that, indeed, the allowed difference is the sole control required for proving the existence of a smooth optimal transportation map G pushing μ_0 to μ_1. Our present work is essentially an attempt toward such a quantification and a proof of Trudinger's conjecture with $r = 2$.
The outline of the paper is as follows. In the sequel of the Introduction, we set up our approach of the regularity problem for optimal transportation maps, a PDE approach, *via* the so-called continuity method [21]. We further state two theorems, our main results, and infer from them several regularity corollaries presented at once with their proofs. The main theorems are proved respectively in Sections 1 and 2. For the reader's convenience, we also provide some auxiliary material required in our proofs adapted from [31] (Appendix A) and [33] (Appendix B), as well as a folklore result mentioned above in the covering spaces setting (Appendix C).

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The continuity method

The optimal map G has the following special form (with obvious notations relative to the metric g):

$$\forall m \in M_n, G(m) = \exp_m(\operatorname{grad}_m u),$$

where the potential-function u, normalized by $\int_{M_n} u \mathrm{dVol} = 0$, is a c-convex function (see [34]). Set $\mathcal{A}$ for the open subset of the Fréchet space

$$C_0^\infty = \{v \in C^\infty(M_n), \int_{M_n} v \ \mathrm{dVol} = 0\}$$

consisting of those functions v such that the map $\exp(\operatorname{grad} v)$ is a diffeomorphism of M_n to itself. One can readily verify that, for each $(v, m) \in \mathcal{A} \times M_n$, the smoothness of $\exp(\operatorname{grad} v)$ requires that the closed geodesic segment

$$\{\exp_m(t \operatorname{grad}_m v), t \in [0, 1]\}$$

does not cross the cut-locus of m (henceforth denoted by Cut_m); in particular, $|\operatorname{grad} v(m)|$ stays bounded above strictly by the diameter of (M_n, g). Fixing the metric g and the smooth positive measure μ_0, let us consider the nonlinear second order differential operator given by:

$$v \in \mathcal{A} \to F(v) := [\rho(v) - \rho_0] \in C_0^\infty, \text{ with } \rho(v) := \frac{d}{\mathrm{dVol}}[\exp(\operatorname{grad} v)_\# \mu_0]$$

(Radon-Nikodym derivative). The operator F is elliptic of Monge–Ampère type and it is a local diffeomorphism which is one-to-one (hence a diffeomorphism) onto its image [16] (see also [17, Remark 6] for an *Erratum* of the proof of the second part of [16, Proposition 3]). Proving that the above optimal map G is smooth thus amounts to proving that F is onto C_0^∞. To do so, given an arbitrary measure μ_1 as above, one may use the continuity method as in [16, p.158] and consider, for $t \in [0, 1]$, the solution $u_t \in \mathcal{A}$ of the pointwise equation expressing the optimal mass transportation of μ_0 to $\mu_t := t\mu_1 + (1 - t)\mu_0$, namely:

$$\exp(\operatorname{grad} u_t)_\# \mu_0 = \mu_t \Longleftrightarrow F(u_t) = t(\rho_1 - \rho_0) \ , \tag{1}$$

arguing by connectedness on the subset $\mathcal{T} \subset [0, 1]$ of t's such that there exists a solution $u_t \in \mathcal{A}$. The set $\mathcal{T}$ obviously contains 0 and it is relatively open in $[0, 1]$; granted $\mathcal{T}$ is closed, one infers $\mathcal{T} = [0, 1]$ hence the map F is indeed onto (and

G, smooth). By standard arguments [21, Section 17.4] (using the *concavity* of the Monge–Ampère type operator $v \mapsto f(x, dv, \nabla d\, v) := F(v)$ with respect to the covariant Hessian variable $\nabla d\, v$, where ∇ stands for the Levi–Civita connection of g), the closedness of $\mathcal{T}$ follows from a uniform pinching (independent of $t \in [0,1]$) on the eigenvalues with respect to the metric g of the symmetric tensor $\nabla d\, u_t$. A uniform lower bound on these eigenvalues is already known [16, p.154]; so the smoothness of the optimal transportation map G boils down to carrying out a uniform *upper* bound on them.

The first genuinely interior bound of that sort (previous bounds would require affine boundary-value data [36, pp.73-76], they were thus never really *interior*) was recently derived by Ma, Trudinger and Wang [33, Theorem 4.1] dealing, in some open subset Ω of $\mathbb{R}^n$, with elliptic Monge–Ampère equations of the form:

$$\det\left[\mathfrak{A}_{ij}(x, v, dv) + \frac{\partial^2 v}{\partial x^i \partial x^j}\right] = \mathfrak{B}(x, v, dv) > 0$$

where $\mathfrak{A}_{ij} = \mathfrak{A}_{ij}(x, z, p)$ is a $n \times n$ symmetric matrix field on $J^1\Omega$ (first jet space). If v is a solution such that the so-called *strict regularity* condition [39] holds, namely (using Einstein's summation convention):

$$(2)\ \exists\theta > 0, \forall(\xi, \nu) \in T_x\Omega \times T_x^*\Omega \text{ with } \nu(\xi) = 0,\ -\frac{\partial^2 \mathfrak{A}_{ij}}{\partial p_k \partial p_l}\xi^i\xi^j\nu_k\nu_l \geq \theta|\xi|^2|\nu|^2$$

on the subset $\{[x, v(x), dv(x)], x \in \Omega\} \subset J^1\Omega$, they derived an upper bound on the eigenvalues of the symmetric matrix $\left(\mathfrak{A}_{ij}(x, v, dv) + \partial^2_{ij} v\right)$ in terms of the constant θ, of the $C^1(\overline{\Omega})$-norm of v, the C^2-norms of $(\mathfrak{A}_{ij})$ and B, and the distance of the point $x \in \Omega$ to the boundary $\partial\Omega$ (see [33, 39]).
In local charts of M_n, equation (1) reads like a Monge–Ampère equation of the above form with a matrix field $\mathfrak{A}_{ij}(x, dv)$ independent of the v variable (see [16, 31] and Appendix B below). Specifically, in a generic chart x of M_n, the matrix $(\mathfrak{A}_{ij})$ which occurs for equation (1) is given by:

$$(3) \qquad \forall(v, m) \in \mathcal{A} \times M_n,\ \left(\mathfrak{A}_{ij}(x, dv) + \partial^2_{ij} v\right) dx^i \otimes dx^j = \mathrm{Hess}^{(c)}(v)(m)\ ,$$

where $x = x(m)$, and $\mathrm{Hess}^{(c)}(v)$ denotes the c-Hessian of v, namely the covariant symmetric 2-tensor field defined by:

$$\mathrm{Hess}^{(c)}(v)(m) := [\nabla d\, c(., q)]|_{[m, \exp(\mathrm{grad}\, v)(m)]} + \nabla d\, v(m)\ ,$$

which is known to be positive definite on M_n for each $v \in \mathcal{A}$ [16, Proposition 3][17, Remark 6]. From this definition, we see that the local quantity:

$$A_{ij}(x, \nabla_x v) := \mathfrak{A}_{ij}(x, dv) + \Gamma^k_{ij}(x)\partial_k v$$

is actually intrinsic, hence globally defined (here the Γ^k_{ij}'s stand as usual for the Christoffel symbols of g in the chart x (*cf. infra*) and $\nabla_x v := T_m x(\mathrm{grad}_m\, v)$ with $x = x(m)$, stands for the local expression of the gradient of v). Indeed, we have:

$$(4) \qquad A_{ij}(x, \nabla_x v) dx^i \otimes dx^j \equiv [\nabla d\, c(., q)]|_{[m, \exp(\mathrm{grad}\, v)(m)]}\ ,$$

and this is the quantity which we will consider below (see (9)) in place of the Ma–Trudinger–Wang local quantity $\mathfrak{A}_{ij}(x, dv)dx^i \otimes dx^j$. Importantly, in that

context, it follows from (4) that the left-hand side of inequality (2) is also intrinsic; it is sometimes called a 'cost-sectional curvature' [31] (or c-curvature, for short). An intrinsic definition of it, is given below (see (8)). More deeply, the fact that the c-curvature depends on the metric g only through the cost-function $c = \frac{1}{2}d_g^2$, as written in [33], was recently interpreted geometrically [28].
Let us say that **condition (2) holds uniformly for equation (1)**, whenever this condition bearing on the matrix field $\mathfrak{A}_{ij}(x, dv)$ given by (3), evaluated at $[m, \exp(\operatorname{grad} u_t)(m)]$, holds at each point $m \in M_n$ with a constant $\theta > 0$ independent of $(m,t) \in M_n \times [0,1]$. Assuming it does (*cf. infra*), the Ma–Trudinger–Wang interior estimate will be shown (in Appendix B) to imply an upper bound on the eigenvalues of the tensor $\mathrm{Hess}^{(c)}(u_t)(m)$. Let us emphasize here that the latter may not be enough to infer an upper bound on $\nabla d\, u_t$. Indeed, on the standard n-sphere, $(n-1)$ eigenvalues of $[\nabla d\, c(., q)]|_{[m,\exp(\operatorname{grad} u_t)(m)]}$ are equal to: $|\operatorname{grad} u_t| \cot(|\operatorname{grad} u_t|)(m)$, hence they diverge to $-\infty$ as $|\operatorname{grad} u_t|(m)$ tends to π, or else, as the image-point of m by $\exp(\operatorname{grad} u_t)$ gets close to a conjugate point of m (its antipode, here). The latter occurence was ruled out in [18]. It enabled Loeper to complete the proof of the smoothness of G after checking the strict regularity condition (2) on the standard sphere [31].
Here, we wish to investigate along the same lines the trickier case of a metric g with variable curvature.

Main results; corollaries and their proofs

Before stating our results, loosing no generality, let us scale the metric g so that its sectional curvature K satisfies:

$$(5) \qquad \min_{M_n} K = 1 \ .$$

Remark 1 For later use, let us record the consequences of the normalization (5) for the geometry and topology of M_n. By Myers theorem [13], it implies:

$$(6) \qquad D := \operatorname{diam}(M_n) \leq \pi$$

and $\pi_1(M_n)$ is finite (setting henceforth $\operatorname{diam}(S)$ for the diameter of a subset $S \subset M_n$ measured in M_n with the distance d_g). Let us set:

$$\eta_M := \left(1 - \frac{D}{\pi}\right) \in [0,1) \ .$$

If $\pi_1(M_n)$ is not trivial, the topology creates a gap for η_M; specifically, the Grove–Shiohama diameter sphere theorem [24] implies: $\eta_M \geq \frac{1}{2}$. If M_n is simply connected, the Toponogov maximal diameter theorem [13, p.110] implies $\eta_M > 0$ unless (M_n, g) is isometric to the standard unit n-sphere, and no gap occurs anymore (as shown by the example of an ellipsoid, see Remark 3 below).

The (open) geodesic ball of radius r centered at $m \in M_n$ will be denoted by $B(m,r)$ and the volume of a Borel subset $S \subset M_n$ for the Lebesgue measure dVol, by $\mathrm{Vol}(S)$.
In section 1 below, we will prove an extension of the result of [18] required for implementing the Ma–Trudinger–Wang estimate on simply connected manifolds (see Remark 3):

Theorem 1 *Assume that the manifold M_n is simply connected and that the sectional curvature of the metric g (normalized by (5)) satisfies: $K < 1.44$. Setting $\varepsilon := 1 - \frac{1}{\sqrt{\max_{M_n} K}} < \frac{1}{6}$ and*

$$C_1 := \sup_{\rho \in [0, \frac{1}{6}], q \in M_n} \rho^{-n/2} \frac{\mathrm{Vol}\,[B(q, 5\pi\rho)]}{\mathrm{Vol}\,\left[B(q, D\sqrt{\rho})\right]} \ ,$$

assume on g the further sectional curvature pinching condition: $\varepsilon^{n/2} C_1 < 1$. For $t \in [0,1]$, set $\rho_t := \dfrac{d\mu_t}{\mathrm{dVol}}$. If the measures μ_0, μ_1 satisfy the inequality:

$$\frac{\max_{[0,1]\times M_n} \rho_t}{\min_{M_n} \rho_0} < \frac{1}{C_1 \ \eta^{n/2}} \tag{7}$$

for some $\eta \in \left(\varepsilon, \dfrac{1}{6}\right)$, then:

$$|\operatorname{grad} u_t| \leq (1-\eta) D \ ,$$

and

$$\forall m \in M_n, \ d_g[\exp(\operatorname{grad} u_t)(m), \mathrm{Cut}_m] \geq (\eta - \varepsilon)\pi.$$

Section 2 will be devoted to proving a fairly general c-curvature estimate on compact positively curved manifolds (Theorem 2 below), essential for any subsequent proof of the regularity of the optimal transportation map G. We require further notations. We set Cut for the closed subset of TM_n defined by:

$$\mathrm{Cut} = \{(m,v) \in M_n \times T_m M_n, \ \exp_m(v) \in \mathrm{Cut}_m\} \ ,$$

and consider the open connected component of $TM_n \setminus \mathrm{Cut}$ containing the zero section, let us denote it here (for convenience) by:

$$\mathrm{NoCut} := \{(m,v), \forall t \in [0,1] \text{ and } (m, tv) \notin \mathrm{Cut}\} \ ,$$

which thus satisfies: $\partial(\mathrm{NoCut}) \subset \mathrm{Cut}$. For $\eta \in (0,1)$, we also set:

$$\mathrm{NoCut}_\eta := \{(m,v) \in \mathrm{NoCut}, \ |v| \leq (1-\eta)\pi\} \ .$$

Remark 2 As already pointed out, for each $(u,m) \in \mathcal{A} \times M_n$, the couple $(m, \operatorname{grad}_m u)$ must lie in NoCut. However, *a priori* estimates on the solutions u_t of equation (1) will require more, namely that the image-point $\exp_m(\operatorname{grad}_m u_t)$ stays uniformly away from the first conjugate point of m on the corresponding geodesic, and this will be checked below *via* a comparison device with the (constant curvature 1) spherical case. The reader may anticipate that, conceivably, it will require the existence of some uniform $\eta > 0$ such that $|\operatorname{grad}_m u_t| \leq (1-\eta)\pi$ and, from Remark 1, that the simply connected case will be the only difficult one. In the latter case, though, Klingenberg's theorem [30] shows that, even though η may get small, any point (m,v) in NoCut_η will stay uniformly away from Cut, provided the curvature is sufficiently pinched. So much for motivating the notation NoCut_η.

Given $(m_0, v_0) \in \mathrm{NoCut}$ and two orthogonal unit vectors $(\xi, \nu) \in (T_{m_0}M_n)^2$, let us define intrinsically the associated *c-curvature* by:

$$\mathcal{C}(m_0, v_0)(\xi, \nu) := -\mathrm{D}d[v \mapsto A(m_0, v)(\xi)]|_{v=v_0}(\nu, \nu) \ , \tag{8}$$

where D stands for the canonical flat connection in $T_{m_0}M_n$ and $\xi \mapsto A(m_0, v)(\xi)$ stands for the quadratic form on $T_{m_0}M_n$ given for $(m_0, v) \in \mathrm{NoCut}$ by:

$$A(m_0, v)(\xi) = \nabla d[m \mapsto c(m, \exp_{m_0}(v))]|_{m=m_0}(\xi, \xi) \ . \tag{9}$$

By formal analogy with the expressions occuring in the spherical case [18, 31], let us set (using, of course, on $T_{m_0}M_n$ the norm defined by g_{m_0}):

$$\overline{A}(m_0, v)(\xi) := |\xi|^2 - (1 - |v| \cot |v|) \left[|\xi|^2 - \frac{(g_{m_0}(\xi, v))^2}{|v|^2} \right] ,$$

and define $\overline{\mathcal{C}}(m_0, v_0)(\xi, \nu)$ by formula (8) computed with $\overline{A}(m_0, v)(\xi)$ instead of $A(m_0, v)(\xi)$. We will require the latter calculation (first treated in [31]); for convenience, it is provided in Appendix A below.

Finally, we set Riem for the Riemann curvature tensor of the metric g, viewed as an endomorphism valued 2-form on M_n and, given vector fields U, V, W, we write $\mathrm{Riem}(U, V)W$ for the resulting vector field. It is convenient to define a further tensor of the former sort, namely:

$$\mathrm{Cur}_1(U, V)W := g(W, V)U - g(W, U)V \ .$$

Anytime a metric has constant curvature $K = 1$, it satisfies: $\mathrm{Riem} = \mathrm{Cur}_1$. We set Scal for the scalar curvature of g and recall the definition of the *concircular curvature* tensor [4]:

$$\mathrm{Concirc} := \mathrm{Riem} - \frac{\mathrm{Scal}}{n(n-1)} \mathrm{Cur}_1 \ .$$

In dimension 2, this tensor identically vanishes; when $n > 2$, its vanishing is equivalent for g to having constant curvature.

We further set $\|.\|_{C^2(M_n,g)}$ for the C^2-norm of tensor fields on M_n, calculated with the metric g and its Levi–Civita connection ∇. Dealing with the various estimates derived in Section 2, we will say that a constant is "under control" whenever it only depends on: the dimension n, $\mathrm{diam}(M_n)$, the metric tensor g and $\|\mathrm{Riem}\|_{C^2(M_n,g)}$. Actually, due to the curvature assumptions made on (M_n, g), each constant under control C occuring in the proofs below will be some universal function of the sole dimension n (with polynomial growth in the variable $\sqrt{n}$, *cf.* Remark 6).

The following result provides a curvature control in terms of which the c-curvature can be bounded below, thus quantifying and proving Trudinger's conjecture:

Theorem 2 *Let (M_n, g) be a compact connected n-dimensional Riemannian manifold satisfying (5) and*

$$\|\mathrm{Riem} - \mathrm{Cur}_1\|_{C^2(M_n,g)} \leq \delta \ , \tag{10}$$

for some real $\delta > 0$. Let $(m_0, v_0) \in \mathrm{NoCut}$; so $|v_0| = (1-\eta_0)\pi$ for some $\eta_0 \in (0,1]$. Assume δ is small enough such that:

$$2\sqrt{n-1}\frac{|v_0|}{\sin|v_0|}\delta \leq \frac{1}{2} \ . \tag{11}$$

There exists a constant $C_2 \geq 1$ under control (thus independent of $(m_0, v_0, \eta_0, \delta)$) such that, for each couple of orthogonal unit vectors $\xi \perp \nu$ in $T_{m_0}M_n$, the following inequality holds:

$$\left|\mathcal{C}(m_0, v_0)(\xi,\nu) - \overline{\mathcal{C}}(m_0, v_0)(\xi,\nu)\right| \leq C_2 \ \frac{\delta}{\eta_0^4} \ . \tag{12}$$

It is a standard exercise to verify that the curvature statement (10) can be written equivalently as follows (with another constant δ of same order):

$$\|\mathrm{Concirc}\|_{C^2(M_n,g)} \leq \delta, \text{ if } n > 2,$$

or:

$$\|K-1\|_{C^2(M_n,g)} \leq \delta, \text{ if } n = 2;$$

we will use below the more convenient form (10).

We are now in position to derive a smoothness result, namely:

Corollary 1 *Let (M_n, g) be a compact simply connected n-dimensional Riemannian manifold satisfying (5) and (10) with δ small enough such that:*

$$1 - \frac{1}{\sqrt{1+\delta}} < \frac{1}{C_1^{2/n}} \tag{13}$$

(where C_1 is the constant defined in Theorem 1) and:

$$\delta < \min\left[\frac{5}{6^4\pi^2 C_2}, \frac{1}{24\sqrt{n-1}}\right] \tag{14}$$

(where C_2 is the constant occuring in (12)). Let (μ_0, μ_1) be smooth positive Borelian measures on M_n of same total mass satisfying (7) for some $\eta \in \left(\varepsilon, \frac{1}{6}\right)$ with $\varepsilon = 1 - \frac{1}{\sqrt{\max_{M_n} K}}$. Assume furthermore that η is large enough such that the following inequalities hold:

$$\delta \leq \frac{1}{4\sqrt{n-1}} \ \eta \ , \tag{15}$$

$$\delta < \frac{1}{\pi^2 C_2}\eta^3(1-\eta) \ . \tag{16}$$

Then the optimal transportation map G (pushing μ_0 to μ_1) is smooth.

Here, the requirement (14) implies (16) and (15) when $\eta = \frac{1}{6}$, and the inequality $\varepsilon = 1 - \frac{1}{\sqrt{\max_{M_n} K}} < \frac{1}{6}$.

In the particular case of an ovaloid Σ_f in $\mathbb{R}^{n+1}$ represented as a radial graph over the unit sphere: $m \in \mathbb{S}^n \to M \in \Sigma_f$ with $\overrightarrow{OM} = e^{f(m)}\overrightarrow{Om}$, the curvature assumptions (13)(15)(16) which, together with condition (7) on the measures μ_0, μ_1, yield strict regularity (in Trudinger's sense [39]) for equation (1), amount to smallness conditions on the C^4-norm of the function f.

Proof. Condition (10) implies $K \leq 1 + \delta$; so, the two pinching conditions of Theorem 1 hold, respectively due to (14) and (13). Using the continuity method and fixing $t \in \mathcal{T}$ (*cf. supra*), we may thus apply Theorem 1 to u_t and conclude that the section $\operatorname{grad} u_t$ of TM_n ranges in NoCut_η. Now we wish to apply Theorem 2 at (m_0, v_0) with $v_0 = \exp(\operatorname{grad} u_t)(m_0)$. We may do so because (15) implies condition (11). Fixing an arbitrary couple of orthogonal unit vectors $\xi \perp \nu$ in $T_{m_0}M_n$, inequality (12) implies:

$$\mathcal{C}(m_0, v_0)(\xi, \nu) - \overline{\mathcal{C}}(m_0, v_0)(\xi, \nu) \geq -C_2 \, \frac{\delta}{\eta_0^4}$$

where $\eta_0 \in [\eta, 1]$ is given by $|\operatorname{grad} u_t|(m_0) = (1 - \eta_0)\pi$. Combining it with the spherical case inequality:

$$\overline{\mathcal{C}}(m_0, v_0)(\xi, \nu) \geq \frac{1}{\pi^2} \max\left(1, \frac{1 - \eta_0}{\eta_0}\right) \tag{17}$$

(proved in Appendix A below), we get the lower bound: $\mathcal{C}(m_0, v_0)(\xi, \nu) \geq \theta_0$ with

$$\theta_0 = \frac{1}{\pi^2} \max\left(1, \frac{1 - \eta_0}{\eta_0}\right) - C_2 \, \frac{\delta}{\eta_0^4} \; .$$

We can improve this bound by writing

$$\theta_0 = \frac{1}{\eta_0^4} \left\{ \frac{1}{\pi^2} \max[\eta_0^4, \eta_0^3(1 - \eta_0)] - C_2 \; \delta \right\}$$

and by noting that the map

$$\eta_0 \in [\eta, 1] \to \max[\eta_0^4, \eta_0^3(1 - \eta_0)]$$

is increasing, equal to $\eta^3(1 - \eta)$ for $\eta_0 = \eta < \frac{1}{6}$; we thus find:

$$\eta_0^4 \theta_0 \geq \theta := \frac{1}{\pi^2} \eta^3 (1 - \eta) - C_2 \; \delta \; .$$

Under assumption (16), the latter right-hand side is *strictly positive* hence we obtain for the c-curvature the uniform lower bound:

$$\mathcal{C}(m_0, v_0)(\xi, \nu) \geq \theta > 0 \; . \tag{18}$$

In other words, the strict regularity condition (2) holds uniformly for equation (1). The Ma–Trudinger–Wang interior estimate [33] thus provides an upper bound on the eigenvalues with respect to g of the covariant symmetric 2-tensor $\mathrm{Hess}^{(c)}(u_t)(m)$ (see Appendix B). A uniform upper bound on $\nabla d\, u_t$ follows, due to Theorem 1, which implies that (setting UM_n for the unit-sphere bundle) the function:

$$(m, \xi) \in UM_n \to A(m, \operatorname{grad}_m u_t)(\xi)$$

is bounded below uniformly with respect to $t \in [0, 1]$ (see *e.g.* [15, Lemma 2.3]). As explained above, it yields the closedness of the set $\mathcal{T}$ of deformation parameters t for which the continuity equation (1) admits a solution $u_t \in \mathcal{A}$ (*cf. supra*). So $\mathcal{T} = [0, 1]$ and the optimal transportation map $G = \exp(\operatorname{grad} u_1)$ is smooth, as desired.

Remark 3 Unless (M_n, g) is isometric to the standard unit sphere, the constant η_M introduced in Remark 1 is strictly positive. However, its value depends on the curvature pinching parameter δ and may vanish with him in such a way that condition (16) of Corollary 1 no longer holds with $\eta = \eta_M$. Indeed, if we take for (M_n, g) the ellipsoid of revolution of $\mathbb{R}^3$ given by:

$$\frac{x^2 + y^2}{r^2} + z^2 = 1, \text{ with } r < 1,$$

then (5) is satisfied and we find $\max_{M_2} K = \frac{1}{r^4}$ hence $\delta \geq \frac{1}{r^4} - 1$, while the expansion of the right-hand side of the inequality:

$$1 - \eta_M = \frac{D}{\pi} \geq \frac{1}{\pi} \int_{-1}^{1} \sqrt{1 + \frac{r^2 z^2}{1 - z^2}}\, dz$$

as $r \to 1$ yields: $\eta_M \leq 1 - r + o(1 - r)$. In particular, indeed, η_M vanishes as $r \uparrow 1$ *i.e.* as the ellipsoid approaches the unit-sphere. Besides, the ratio $\frac{\delta}{\eta_M^3}$ blows up at least like $(1 - r)^{-2}$ as $r \uparrow 1$ and condition (16) with $\eta = \eta_M$, which would serve to check the positivity of the c-curvature of our ellipsoid in the absence of a precise calculation of it, fails. This fact explains why, in the simply connected case, we require a condition like (7) on the data (μ_0, μ_1).
Of course, it would be important (although quite lengthy and outside our present scope) to investigate the sign of the leading blowing-up term which occurs in the expression of the c-curvature of our ellipsoid of revolution at the point $(m_0, v_0) = [(0, 0, 1), (0, 0, -(1 - \eta)\pi)]$ as $r \uparrow 1$ and $\eta \downarrow 0$.

Interestingly, one can do without any condition imposed on the measures provided the manifold M_n has nontrivial topology:

Corollary 2 *Let (M_n, g) be a compact connected n-dimensional Riemannian manifold satisfying (5) and (10). Assume $\pi_1(M_n)$ is nontrivial and δ is small enough such that:*

$$\delta < \min\left(\frac{1}{\pi^2 C_2}\eta_M^4, \frac{1}{4\sqrt{n-1}}\eta_M\right) \tag{19}$$

(where C_2 is the constant occuring in (12)). Then the optimal transportation map G (pushing μ_0 to μ_1) is smooth.

Proof. On the one hand, from the nontrivial topology and (5), we have: $D \leq \frac{\pi}{2}$ [24]. On the other hand, recalling $K \leq 1 + \delta$, the Rauch comparison theorem [13, p.29][10, p.215] readily yields for the conjugate radius the lower bound:

$$\text{conj}(M_n) \geq \frac{\pi}{\sqrt{1 + \delta}}\ .$$

Furthermore, since $C_2 \geq 1$ and $\eta_M < 1$, inequality (19) implies $\delta < \frac{1}{\pi^2}$ hence $\text{conj}(M_n) \geq \pi - \frac{1}{2\pi}$; in particular, we get:

$$\text{conj}(M_n) > \text{diam}(M_n).$$

It follows that the exponential map must be nonsingular on Cut. Besides, since $\eta_M \geq \frac{1}{2}$, arguing as above now with $\eta = \eta_M$ such that $\max[\eta^4, \eta^3(1-\eta)] = \eta^4$, condition (19) combined with Theorem 2 implies that the c-curvature of M_n is positive. Corollary 2 now follows from the result (v) of Loeper–Villani [32] mentionned at the beginning of the introduction. Alternatively, using the continuity method and fixing $t \in \mathcal{T}$, we simply note that, for each $m \in M_n$, the inverse of the tangent map

$$d(\exp_m)(\operatorname{grad}_m u_t) : T_m M_n \to T_{\exp_m(\operatorname{grad}_m u_t)} M_n$$

has its g-norm bounded above by a constant independent of $m \in M_n$ and $t \in \mathcal{T}$ (equal to $\sqrt{1+(n-1)\pi^2}$ as shown by the inequality (71) below, read here with $|v_0| = \frac{\pi}{2}$). This key-estimate enables one to apply the Ma–Trudinger–Wang device (see Appendix B) and conclude as above.

Back to the simply connected case, Corollary 2 yields an alternative (symmetry) condition on the given measures, sufficient for the existence of a smooth optimal transport:

Corollary 3 *Let $(\widetilde{M_n}, \widetilde{g})$ be compact simply connected satisfying (5) and (10). Let $(\widetilde{\mu_0}, \widetilde{\mu_1})$ be smooth positive Borelian measures on $\widetilde{M_n}$ of same total mass, invariant under a non-trivial subgroup of isometries Γ acting on $\widetilde{M_n}$ in a totally discontinuous way. Set (M_n, g) for the quotient manifold and*

$$\eta_M = 1 - \frac{\operatorname{diam}(M_n)}{\pi} \ ;$$

assume that the pinching constant δ occuring on $\widetilde{M_n}$ for (10) is small enough such that (19) holds. Then the optimal transportation map $\widetilde{G}$ (pushing $\widetilde{\mu_0}$ to $\widetilde{\mu_1}$) is smooth.

Proof. Set $p : \widetilde{M_n} \to M_n$ for the natural (covering space) projection and r for its degree (fiber cardinal). From the Γ-invariance of the measures, there exists a couple of smooth positive Borelian measures (μ_0, μ_1) on M_n such that $r\mu_i = p_{\#}\widetilde{\mu_i}$ for each $i \in \{0,1\}$. By naturality and under our assumption on δ, the manifold (M_n, g) fulfills the hypothesis of Corollary 2. Accordingly, let $G = \exp_g(\operatorname{grad}_g u) : M_n \to M_n$ be the smooth optimal transportation map pushing μ_0 to μ_1. The map $\widetilde{G} = \exp_{\widetilde{g}}(\operatorname{grad}_{\widetilde{g}} p^* u)$ satisfies $\widetilde{G}_{\#}\widetilde{\mu_0} = \widetilde{\mu_1}$ (a general fact, see Appendix C); it is a smooth optimal transportation map for our original data, the unique one [34, 16].

1 Distance from cut-locus

This section is devoted to the proof of Theorem 1. In the next two subsections, we return to a compact connected n-dimensional Riemannian manifold (M_n, g) with no particular curvature assumption. We will get back to assumption (5) subsequently.

1.1 2-monotonicity of optimal maps

Recall that a map $\Phi : M_n \to M_n$ is called 2-monotonous with respect to the geodesic distance d_g if it satisfies the following: $\forall (m_1, m_2) \in M_n^2$,

$$d_g^2[m_1, \Phi(m_1)] + d_g^2[m_2, \Phi(m_2)] \leq d_g^2[m_1, \Phi(m_2)] + d_g^2[m_2, \Phi(m_1)] \ . \tag{20}$$

For completeness, we will prove here the continuous version of a 2-monotonicity lemma which would hold almost-everywhere under weaker assumptions – not required below – as in [18, 31]. It is a particular case of a property (called c-cyclicity) valid in a very general context [20, Theorem 2.7].

Lemma 1 *For each couple of continuous positive Borelian measures (μ, ν) with same total mass, if the optimal transportation map G such that $G_{\#}\mu = \nu$ is continuous, it is 2-monotonous.*

Proof. We adapt the argument of [18, Lemma 1]. Pick two distinct points $(m_1, m_2) \in M_n^2$ and fix a small real $r > 0$. Set $B_{1r} = B(m_1, r)$ and take $\rho > 0$ such that the ball $B_{2\rho} = B(m_2, \rho)$ satisfies: $\mu(B_{1r}) = \mu(B_{2\rho})$. By [19, Theorem 8.6], there exists a μ-preserving diffeomorphism $\varphi_r : B_{1r} \to B_{2\rho}$, out of which we may define a μ-preserving map $\psi_r : M_n \to M_n$ as follows:

$$\psi_r = \varphi_r \text{ on } \overline{B_{1r}};\ \psi_r = \varphi_r^{-1} \text{ on } \overline{B_{2\rho}};\ \psi_r = \mathrm{Id}\ \text{ elsewhere.}$$

As in [18, p.301], write:

$$\frac{1}{\mu(B_{1r})}[\mathcal{C}(G) - \mathcal{C}(G \circ \psi_r)] \leq 0$$

and let $r \to 0$ to get the desired conclusion.

1.2 Big-crunch argument

Let us denote by $\mathcal{N}_r(S)$ the open r-neighborhood of a subset $S \subset M_n$, that is, the set $\{p \in M_n, \exists q \in S, d_g(p, q) < r\}$.

Proposition 1 *Assume the following condition on the manifold (M_n, g): for $s > 0$ small, there exists a positive increasing function $s \mapsto f(s)$ with $\lim_{s \downarrow 0} f(s) = 0$ such that: $\exists \eta_0 > 0, \forall \eta \in (0, \eta_0), \forall (m, q) \in M_n \times \mathrm{Cut}_m$,*

$$\frac{\mathrm{Vol}[B(q, 4D\sqrt{\eta}) \cap \mathcal{N}_{3D\eta}(\mathrm{Cut}_m)]}{\mathrm{Vol}[B(q, D\sqrt{\eta})]} \leq f(\eta) \ . \tag{21}$$

Take $\eta_0 \leq \frac{1}{6}$ with no loss of generality. Given two positive continuous Borelian measures $\mu_0 = \rho_0 \mathrm{dVol}$ and $\mu_1 = \rho_1 \mathrm{dVol}$ on M_n with same total mass and $\eta \in (0, \eta_0)$ such that:

$$\frac{max_{M_n}\rho_1}{min_{M_n}\rho_0} < \frac{1}{f(\eta)} \ ,$$

the optimal transportation map G pushing μ_0 to μ_1, if it is continuous, satisfies:

$$\forall m \in M_n,\ d_g[m, G(m)] \leq (1 - \eta)D \ .$$

Proof. By continuity, the set $\{d_g[m, G(m)], m \in M_n\} \subset \mathbb{R}$ is connected and closed; we prove: $\max_{M_n} d_g[m, G(m)] \leq (1-\eta)D$, arguing by contradiction. Set $d = \frac{d_g}{D}$ and fix $m \in M_n$ such that $d[m, G(m)] > 1-\eta$. Let $[m, m']$ be a maximal geodesic segment containing $G(m)$. So $m' \in \mathrm{Cut}_m$ and $d[G(m), m'] < \eta$. Consider the open geodesic ball $B(m', \sqrt{\eta})$. By Lemma 1, for any $p \in B(m', \sqrt{\eta})$, we have:

$$d^2(m, G(m)) + d^2(p, G(p)) \leq d^2(m, G(p)) + d^2(p, G(m)).$$

Using the triangle inequality, and since $\eta < \frac{1}{6}$, we get the lower bound:

$$d^2(m, G(m)) - d^2(p, G(m)) \geq d^2(m, G(m)) - [d(p, m') + d(m', G(m))]^2 \geq (1-3\eta)^2,$$

which, combined with the 2-monotonicity inequality, yields:

$$(1-3\eta)^2 + d^2(p, G(p)) \leq d^2(m, G(p)).$$

On the one hand, since $d(m, G(p)) \leq 1$, it implies $d(p, G(p)) \leq 3\sqrt{\eta}$, hence, by the triangle inequality: $d(m', G(p)) < 4\sqrt{\eta}$; on the other hand, we infer: $d(m, G(p)) > 1-3\eta$. Altogether, we thus have:

$$G(p) \in \mathcal{N}_{3D\eta}(\mathrm{Cut}_m) \cap B(m', 4D\sqrt{\eta}) \ ;$$

in other words:

$$G[B(m', D\sqrt{\eta})] \subset \mathcal{N}_{3D\eta}(\mathrm{Cut}_m) \cap B(m', 4D\sqrt{\eta}) \ .$$

Since $G_\# \mu_0 = \mu_1$, the preceding inclusion implies:

$$(22) \quad \min_{M_n} \rho_0 \ \ \mathrm{Vol}[B(m', D\sqrt{\eta})] \leq \max_{M_n} \rho_1 \ \ \mathrm{Vol}\left[\mathcal{N}_{3D\eta}(\mathrm{Cut}_m) \cap B(m', 4D\sqrt{\eta})\right]$$

which contradicts the assumption.

1.3 Geometric estimates

In case M_n is simply connected, let us show that condition (21) holds with η reasonably small, provided the curvature of g, normalized by (5), is sufficiently pinched. We denote below by $\mathrm{inj}(M_n)$ (or i for short) the injectivity radius of the manifold (M_n, g).

Proposition 2 *Assume that M_n is simply connected satisfying (5) and:*

$$(23) \qquad \exists \alpha \in (0,3), \ K < (1+\alpha).$$

Then the following pinching holds for the distance from a generic point to any point of its cut-locus:

$$(24) \qquad \frac{\pi}{\sqrt{1+\alpha}} \leq \mathrm{inj}(M_n) \leq \mathrm{diam}(M_n) \leq \pi \ ,$$

and, setting $\varepsilon := 1 - \frac{1}{\sqrt{1+\alpha}}$, we have for each $m \in M_n$:

$$(25) \qquad \mathrm{diam}(\mathrm{Cut}_m) \leq 2\varepsilon\pi \ .$$

Moreover, for $\alpha < 0.44$ (or else $\varepsilon < \frac{1}{6}$), there exists a constant $C_1 \geq 1$ independent of $\varepsilon \in \left(0, \frac{1}{6}\right)$ such that, for each $(m, q) \in M_n \times \mathrm{Cut}_m$ and each $\eta \in \left(\varepsilon, \frac{1}{6}\right)$, condition (21) holds with $f(\eta) = C_1 \eta^{n/2}$.

Proof. Under condition (23), recalling (6), Klingenberg's theorem [30] implies $i \geq \dfrac{\pi}{\sqrt{1+\alpha}}$, proving (24).
In order to prove (25), we fix $m \in M_n, (p, q) \in \mathrm{Cut}_m^2$ and consider the hinge $\widehat{pmq}$ forming an angle β at m. Let us consider a comparison hinge $\widehat{\bar{p}\bar{m}\bar{q}}$ in the standard unit-sphere $\mathbb{S}^n$ with: $d_g(m, p) = d_{\mathbb{S}^n}(\bar{m}, \bar{p}), d_g(m, q) = d_{\mathbb{S}^n}(\bar{m}, \bar{q})$ and same angle β at $\bar{m}$. From (24), we have:

$$\forall r \in \{p, q\},\ (1-\varepsilon)\pi \leq d_g(m, r) \leq \pi\ . \tag{26}$$

By Toponogov's theorem [19], we infer: $d_g(p, q) \leq d_{\mathbb{S}^n}(\bar{p}, \bar{q})$. Setting $\bar{m'}$ for the antipodal point of $\bar{m}$ in $\mathbb{S}^n$, the triangle inequality yields:

$$\begin{aligned} d_{\mathbb{S}^n}(\bar{p}, \bar{q}) \leq d_{\mathbb{S}^n}(\bar{p}, \bar{m'}) + d_{\mathbb{S}^n}(\bar{m'}, \bar{q}) &= \pi - d_{\mathbb{S}^n}(\bar{m}, \bar{p}) + \pi - d_{\mathbb{S}^n}(\bar{m}, \bar{q}) \\ &\equiv \pi - d_g(m, p) + \pi - d_g(m, q)\ . \end{aligned}$$

Altogether, recalling (26), we conclude: $d_g(p, q) \leq 2\varepsilon\pi$, proving (25).
As for the final part of Proposition 2, given $(m, q) \in M_n \times \mathrm{Cut}_m$ and $\eta \in \left(\varepsilon, \frac{1}{6}\right)$, we note that the inequality (25) implies:

$$\mathcal{N}_{3D\eta}(\mathrm{Cut}_m) \subset B(q, 5\pi\eta)\ ,$$

hence:

$$\frac{\mathrm{Vol}[B(q, 4D\sqrt{\eta}) \cap \mathcal{N}_{3D\eta}(\mathrm{Cut}_m)]}{\mathrm{Vol}[B(q, D\sqrt{\eta})]} \leq \frac{\mathrm{Vol}[B(q, 5\pi\eta)]}{\mathrm{Vol}[B(q, D\sqrt{\eta})]} \leq C_1 \eta^{n/2}\ ,$$

where C_1 is the constant defined in Theorem 1, by:

$$C_1 := \sup_{\rho \in [0, \frac{1}{6}], q \in M_n} \rho^{-n/2} \frac{\mathrm{Vol}\left[B(q, 5\pi\rho)\right]}{\mathrm{Vol}\left[B(q, D\sqrt{\rho})\right]}\ .$$

Under our curvature pinching assumption, we can estimate the constant C_1 by means of standard volume comparison theorems. Specifically, for $\rho > 0$ small enough (see (27) below), the Bishop inequality [3] (applied with (5)) yields:

$$\mathrm{Vol}[B(q, 5\pi\rho)] \leq \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \int_0^{5\pi\rho} (\sin t)^{n-1} dt\ ,$$

while the Günther inequality [25] (applied with (23)) provides:

$$\mathrm{Vol}[B(q, D\sqrt{\rho})] \geq \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \int_0^{D\sqrt{\rho}} \left[\frac{\sin(\sqrt{1+\alpha}\ t)}{\sqrt{1+\alpha}}\right]^{n-1} dt\ .$$

Combining the two inequalities yields, after some calculations, the upper bound:

$$C_1 \leq \left(\frac{5\pi}{D}\right)^n \frac{1}{(1+\alpha)^{\frac{n}{2}-1}\left[1 - \left(\frac{0{,}04\pi^2}{n+2}\right)\right]}\ .$$

Regarding the size of ρ in this argument, the Bishop–Günther inequalities hold on balls with radius smaller than the injectivity radius. Here, recalling (24), we require:

$$\max(5\pi\rho, \pi\sqrt{\rho}) < \frac{\pi}{\sqrt{1+\alpha}} \ , \tag{27}$$

which, to be consistent with the condition $\eta > \varepsilon = 1 - \frac{1}{\sqrt{1+\alpha}}$ when $\rho = \eta$, implies for α the inequality:

$$\max\left[5\left(1 - \frac{1}{\sqrt{1+\alpha}}\right), \sqrt{1 - \frac{1}{\sqrt{1+\alpha}}}\right] < \frac{1}{\sqrt{1+\alpha}} \ ,$$

satisfied for $\alpha < 0.44$. The latter combined with (27) yields $\rho < \frac{1}{6}$.
Finally, using assumption (5), a lower bound on C_1 follows from the Bishop–Gromov inequality [23, 12, 35, 37] which reads, for $\rho > 0$ small enough:

$$\frac{\mathrm{Vol}[B(q, 5\pi\rho)]}{\mathrm{Vol}[B(q, D\sqrt{\rho})]} \geq \frac{\int_0^{5\pi\rho} (\sin t)^{n-1} dt}{\int_0^{D\sqrt{\rho}} (\sin t)^{n-1} dt} \ ,$$

and which, recalling (6), yields: $C_1 > \dfrac{5^{n+1}}{8}$; in particular $C_1 > 1$ as claimed.

1.4 Completion of the proof of Theorem 1

Under the assumption made on the manifold (M_n, g), Proposition 2 holds. Its final part ensures that assumption (21) of Proposition 1 holds provided $\eta > \varepsilon$. Applying the latter proposition with $\eta \in \left(\varepsilon, \frac{1}{6}\right)$ and with ρ_1 replaced by ρ_t, we get:

$$\forall m \in M_n, \ d_g[m, \exp(\operatorname{grad} u_t)(m)] \leq (1-\eta) D$$

or else: $|\operatorname{grad} u_t| \leq (1-\eta)D$, as desired. To derive the second inequality of Theorem 1, we use the triangle inequality:

$$d_g[\exp(\operatorname{grad} u_t)(m), \mathrm{Cut}_m] \geq d_g(m, \mathrm{Cut}_m) - d_g[m, \exp(\operatorname{grad} u_t)(m)]$$

combined with the preceding one, getting:

$$d_g[\exp(\operatorname{grad} u_t)(m), \mathrm{Cut}_m] \geq i - (1-\eta) D \ ,$$

and we finish the proof using (24).

2 c-curvature estimate

Section 2 is devoted to the proof of Theorem 2; here is the strategy. Fixing $(m_0, v_0) \in \mathrm{NoCut}_\eta$, we may assume $v_0 \neq 0$ with no loss of generality. Indeed, if $v_0 = 0$ (so $\eta_0 = 1$), recalling (5)(10), the c-curvature satisfies:

$$\left|\mathcal{C}(m_0, 0)(\xi, \nu) - \overline{\mathcal{C}}(m_0, 0)(\xi, \nu)\right| \leq \frac{2}{3}\delta$$

since it is equal to $\frac{2}{3}k$ with k the sectional curvature of the manifold at m_0 for the 2-plane defined by (ξ, ν) [31, 28] (see Remark 5 below); so (12) readily holds

with $C_2 = \frac{2}{3}$.
Henceforth we take $|v_0| \neq 0$. We will compute the quadratic form $A(m_0, v)(\xi)$ for $v \in T_{m_0}M_n$ close to v_0 (section 2.2), then differentiate it twice with respect to v at $v = v_0$. Unless the curvature $K \equiv 1$, the expression of $A(m_0, v)(\xi)$ is not an explicit function of v; it is obtained from the value taken at time 1 by the solutions produced by initial data variation along the geodesic which starts from the point m_0 with the velocity v. So we must proceed stepwise, viewing the initial data (m, v) as *parameters* in the Cauchy problem for the geodesic equation; we will differentiate that problem with respect to those parameters, three times successively (sections 2.3 to 2.5). To treat the resulting expressions at each step, we will view them as *perturbations* of the corresponding ones in the spherical case. Finally, putting intermediate quantities together, we will write an expansion of the c-curvature (8) starting out with the spherical expression, and estimate the order of the next term, adjusting the size of the curvature deformation parameter δ and of the, so to say, distance from conjugate-locus parameter η_0 (section 2.6).

2.1 Riemannian tools in Fermi charts

For completeness, let us recall auxiliary tools from Riemannian geometry [2, 22, 35, 37, 38], thus letting again provisionally (M_n, g) be a compact connected n-dimensional Riemannian manifold with no particular curvature assumption. Our sign convention for the Riemann curvature tensor is:

$$\mathrm{Riem}(U, V) = [\nabla_U, \nabla_V] - \nabla_{[U,V]}$$

where $[.,.]$ stands successively for a covariant derivatives commutator and for the Lie bracket of the vector fields U, V. In any local chart $\left(x^1, \ldots, x^n\right)$, setting $\partial_i = \frac{\partial}{\partial x^i}$, the i-th component R^i_{jkl} of the local vector field $[\mathrm{Riem}(\partial_k, \partial_l)\partial_j]$ is thus given by:

$$R^i_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{ks}\Gamma^s_{jl} - \Gamma^i_{ls}\Gamma^s_{jk}$$

where the Γ^i_{jk}'s stand for the Christoffel symbols of the Levi–Civita connection ∇, equal to:

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\partial_k g_{jl} + \partial_j g_{kl} - \partial_l g_{jk}\right), \text{ with } g^{il} g_{lj} \equiv \delta^i_j \ .$$

The sectional curvature tensor is defined by:

$$\mathrm{Sect}(U, V, W, Z) = g[U, \mathrm{Riem}(W, Z)V]$$

and its components, accordingly by $R_{ijkl} = g_{is} R^s_{jkl}$.

Definition 1 (Fermi chart) *Given* $(m_0, v_0) \in \mathrm{NoCut}$, *with* $v_0 \neq 0$, *and an orthonormal basis* $(e_1, \ldots, e_n)$ *of* $T_{m_0}M_n$ *with* $e_n = \dfrac{v_0}{|v_0|}$, *the associated Fermi chart* $x = \left(x^1, \ldots, x^n\right)$ *along the normalized geodesic:*

$$s \in [0, |v_0|] \to c(s) := \exp_{m_0}(s e_n)$$

(the latter will be called 'the axis' of the chart, for short) is defined, after parallel transport of the orthonormal basis $(e_1, \ldots, e_n)$ *along the axis, by:*

$$x(m) = \left(x^1, \ldots, x^n\right) \iff m = \mathcal{F}(x) := \exp_{c(x^n)}\left(\sum_{\alpha=1}^{n-1} x^\alpha e_\alpha\right) .$$

The differential of $\mathcal{F}$ *on* $\{x \in \mathbb{R}^n, x^1 = \ldots = x^{n-1} = 0, 0 \leq x^n \leq |v_0|\}$ *is readily found equal to the identity; so, indeed, with* $(m_0, v_0) \in \mathrm{NoCut}$, *there exists a neighborhood of the axis on which the map* $\mathcal{F}$ *defines a chart.*

Note that, in this definition, we keep the flexibility of rotating all basis vectors at m_0 but the last one e_n.
Along the axis, the geodesic motion: $t \in [0,1] \to \exp_{m_0}(tv_0)$ simply reads: $t \mapsto (0, \ldots, 0, t|v_0|)$, and the chart is *normal* (in particular, Christoffel symbols vanish), meaning:

$$\forall x^n \in [0, |v_0|], \forall i, j, k \in \{1, \ldots, n\},\ g_{ij}(0, x^n) = \delta_{ij},\ \partial_k g_{ij}(0, x^n) = 0,$$

(see *e.g.* [2, 35]). We will require higher order non-intrinsic quantities which become of geometrical significance on the axis; specifically, letting latin indices range in $\{1, \ldots, n\}$, greek indices in $\{1, \ldots, n-1\}$, we will prove the following explicit formulas (of independent interest):

Lemma 2 *The following identities hold on the axis:*

$$\partial_{\alpha\beta} g_{nn} = -2R_{n\alpha n\beta} \quad , \quad \partial_{\alpha\beta} g_{n\gamma} = -\frac{2}{3}(R_{\gamma\alpha n\beta} + R_{\gamma\beta n\alpha}) \ ; \tag{28}$$

$$\partial_\alpha \Gamma^i_{jn} = R^i_{j\alpha n} \quad , \quad \partial_\alpha \Gamma^i_{\beta\gamma} = \frac{1}{3}\left(R^i_{\beta\alpha\gamma} + R^i_{\gamma\alpha\beta}\right) \ ; \tag{29}$$

$$\partial_{\alpha\beta} \Gamma^i_{nn} = \nabla_\alpha R^i_{n\beta n} + \nabla_n R^i_{\beta\alpha n} \ ; \tag{30}$$

$$\partial_{\alpha\beta} \Gamma^n_{n\gamma} = \frac{1}{3}\nabla_n (R^n_{\beta\alpha\gamma} - R^n_{\gamma\beta\alpha}) - \nabla_\alpha R^n_{\gamma n\beta} \ ; \tag{31}$$

$$\begin{aligned} \partial_{\alpha\beta} \Gamma^\lambda_{n\gamma} &= \frac{1}{2}\left(\nabla_\alpha R^\lambda_{\gamma\beta n} + \nabla_\beta R^\lambda_{\gamma\alpha n}\right) \\ &+ \frac{1}{6}\nabla_n \left(R^\lambda_{\alpha\beta\gamma} + R^\lambda_{\beta\alpha\gamma}\right) \ . \end{aligned} \tag{32}$$

Moreover, applying m *times* ∂_n *(axis-derivative) to any of the preceding non-intrinsic left-hand quantities, yields on the axis the* m*-th covariant derivative* ∇^m_n *of the corresponding intrinsic right-hand quantity. For instance:*

$$\partial_n(\partial_\alpha \Gamma^i_{jn}) = \nabla_n R^i_{jkn} \ .$$

A further formula (the one for $\partial_{\alpha\beta}\Gamma^n_{\lambda\gamma}$), only required to implement the Ma–Trudinger–Wang estimate, will be stated and established in Appendix B.

Proof. The first formula of line (28) is routine from the definition. The second one is not; it is obtained by combining the first Bianchi identity with the following Fermi analogue (read with $i = n$) of a key-identity first proved in geodesic polar coordinates by Riemann, namely:

$$\sum_{(\alpha,\beta,\gamma)} \partial_{\alpha\beta} g_{i\gamma} = 0 \ , \tag{33}$$

where $\sum_{(\alpha,\beta,\gamma)}$ means circular summation on (α, β, γ). The proof of (33) is a straightforward adaptation of the one given in [38, chap.4, prop.4] (see Appendix

B, proof of Lemma 16); we will thus omit it. Here, for later use, let us pause and derive yet another identity of the type (33) known to Riemann as well, namely:

$$\sum_{(\alpha,\beta,\gamma)} \partial_{\mu\alpha} g_{\beta\gamma} = 0 \ . \tag{34}$$

We prove it by applying (33) to anyone of its summands, say to $\partial_{\mu\gamma} g_{\alpha\beta}$, which makes the preceding circular sum equal to:

$$\partial_{\mu\alpha} g_{\beta\gamma} - \partial_{\beta\gamma} g_{\mu\alpha} \ .$$

By symmetry, it is thus also equal to:

$$\frac{1}{3} \sum_{(\alpha,\beta,\gamma)} \left(\partial_{\mu\alpha} g_{\beta\gamma} - \partial_{\beta\gamma} g_{\mu\alpha}\right) \ .$$

Now (33) yields equality to $\frac{1}{3} \sum_{(\alpha,\beta,\gamma)} \partial_{\mu\alpha} g_{\beta\gamma}$ proving the desired vanishing. As a by-product of that argument, we get on the axis the further identity:

$$\forall \alpha, \beta, \gamma, \mu \in \{1, \ldots, n-1\}, \ \partial_{\mu\alpha} g_{\beta\gamma} = \partial_{\beta\gamma} g_{\mu\alpha} \ . \tag{35}$$

Back to the proof of Lemma 2, the first formula of line (29) can be routinely verified from the local formula defining the curvature. As regards the second formula, first with $i = n$, direct calculation provides:

$$\partial_\alpha \Gamma^n_{\beta\gamma} = \frac{1}{2}(\partial_{\alpha\beta} g_{n\gamma} + \partial_{\alpha\gamma} g_{n\beta})$$

and the desired formula follows from the second one of line (28). Still for the second formula of line (29), now with $i = \lambda$, the definition of the curvature yields $\partial_\alpha \Gamma^\lambda_{\beta\gamma} = R^\lambda_{\beta\alpha\gamma} + \partial_\gamma \Gamma^\lambda_{\alpha\beta} = R^\lambda_{\gamma\alpha\beta} + \partial_\beta \Gamma^\lambda_{\alpha\gamma}$, hence also:

$$\partial_\alpha \Gamma^\lambda_{\beta\gamma} = \frac{1}{3} \left(R^\lambda_{\beta\alpha\gamma} + R^\lambda_{\gamma\alpha\beta} + \sum_{(\alpha,\beta,\gamma)} \partial_\alpha \Gamma^\lambda_{\beta\gamma} \right) \ .$$

From the latter formula we are done: indeed, the mere definition of the Christoffel symbols provides the equality

$$\sum_{(\alpha,\beta,\gamma)} \partial_\alpha \Gamma^\lambda_{\beta\gamma} = \frac{1}{2} \sum_{(\alpha,\beta,\gamma)} \left(\partial_{\alpha\beta} g_{\lambda\gamma} + \partial_{\alpha\gamma} g_{\lambda\beta} - \partial_{\alpha\lambda} g_{\beta\gamma}\right)$$

the right-hand side of which vanishes by (33) and (34).
The formula of line (30) follows from the first one of line (29) by applying ∂_α to the local expression defining $R^i_{n\beta n}$ and by using the final (obvious) formula of the lemma.
As regards (31), brut calculation yields:

$$\partial_{\alpha\beta} \Gamma^n_{n\gamma} = \frac{1}{2} \partial_{\alpha\beta\gamma} g_{nn} \ , \ \partial_{\alpha\beta} \Gamma^\gamma_{nn} = \frac{1}{2}(2\partial_{n\alpha\beta} g_{n\gamma} - \partial_{\alpha\beta\gamma} g_{nn}) \ .$$

Combining these equalities, we infer: $\partial_{\alpha\beta} \Gamma^n_{n\gamma} = \partial_n(\partial_{\alpha\beta} g_{n\gamma}) - \partial_{\alpha\beta} \Gamma^\gamma_{nn}$, and we conclude by using (30) and the second formula of (28).

For (32), we first compute on the axis: $\nabla_\alpha R^\lambda_{\gamma\beta n} = \partial_{\alpha\beta}\Gamma^\lambda_{n\gamma} - \partial_{n\alpha}\Gamma^\lambda_{\gamma\beta}$ and infer, by symmetry with respect to (α,β), the equality:

$$\partial_{\alpha\beta}\Gamma^\lambda_{n\gamma} = \frac{1}{2}\left[\nabla_\alpha R^\lambda_{\gamma\beta n} + \nabla_\beta R^\lambda_{\gamma\alpha n} + \partial_n(\partial_\alpha \Gamma^\lambda_{\gamma\beta} + \partial_\beta \Gamma^\lambda_{\gamma\alpha})\right] \ .$$

But on the axis, using (33), we readily find: $(\partial_\alpha\Gamma^\lambda_{\gamma\beta} + \partial_\beta\Gamma^\lambda_{\gamma\alpha}) = \partial_{\alpha\beta} g_{\gamma\lambda}$ hence, circular summing on (α,β,γ) the second last equality and using again (33) yields:

$$\sum_{(\alpha,\beta,\gamma)} \partial_{\alpha\beta}\Gamma^\lambda_{n\gamma} = \sum_{(\alpha,\beta,\gamma)} \nabla_\alpha R^\lambda_{\gamma\beta n} \ . \tag{36}$$

Moreover, from the above equality we also get:

$$\begin{aligned}\partial_{\alpha\beta}\Gamma^\lambda_{n\gamma} - \partial_{\beta\gamma}\Gamma^\lambda_{n\alpha} &= \frac{1}{2}[\nabla_\alpha R^\lambda_{\gamma\beta n} + \nabla_\beta R^\lambda_{\gamma\alpha n} \\ &- \nabla_\beta R^\lambda_{\alpha\gamma n} - \nabla_\gamma R^\lambda_{\alpha\beta n} + \partial_n(\partial_\alpha\Gamma^\lambda_{\gamma\beta} - \partial_\gamma\Gamma^\lambda_{\alpha\beta})]\end{aligned}$$

and we recognize that the final right-hand parenthesis is nothing but $\nabla_n R^\lambda_{\beta\alpha\gamma}$. Combining this with a similar calculation for $(\partial_{\alpha\beta}\Gamma^\lambda_{n\gamma} - \partial_{\alpha\gamma}\Gamma^\lambda_{n\beta})$, we find for the left-hand side of (36):

$$\sum_{(\alpha,\beta,\gamma)} \partial_{\alpha\beta}\Gamma^\lambda_{n\gamma} = 3\ \partial_{\alpha\beta}\Gamma^\lambda_{n\gamma} + \text{first covariant derivatives of the Riemann tensor},$$

specifically:

$$\begin{aligned}+\ \frac{1}{2}[\nabla_\beta(R^\lambda_{\alpha\gamma n} - R^\lambda_{\gamma\alpha n}) &+ \nabla_\gamma R^\lambda_{\alpha\beta n} - \nabla_\alpha R^\lambda_{\gamma\beta n} - \nabla_n R^\lambda_{\beta\alpha\gamma} \\ + \nabla_\alpha(R^\lambda_{\beta\gamma n} - R^\lambda_{\gamma\beta n}) &+ \nabla_\gamma R^\lambda_{\beta\alpha n} - \nabla_\beta R^\lambda_{\gamma\alpha n} - \nabla_n R^\lambda_{\alpha\beta\gamma}] \ .\end{aligned}$$

From the latter equality combined with (36), we obtain:

$$\begin{aligned}\partial_{\alpha\beta}\Gamma^\lambda_{n\gamma} &= \frac{1}{3}\left(\sum_{(\alpha,\beta,\gamma)} \nabla_\alpha R^\lambda_{\gamma\beta n}\right) - \frac{1}{6}[\nabla_\beta(R^\lambda_{\alpha\gamma n} - R^\lambda_{\gamma\alpha n}) + \nabla_\gamma R^\lambda_{\alpha\beta n} - \nabla_\alpha R^\lambda_{\gamma\beta n} \\ &- \nabla_n R^\lambda_{\beta\alpha\gamma} + \nabla_\alpha(R^\lambda_{\beta\gamma n} - R^\lambda_{\gamma\beta n}) + \nabla_\gamma R^\lambda_{\beta\alpha n} - \nabla_\beta R^\lambda_{\gamma\alpha n} - \nabla_n R^\lambda_{\alpha\beta\gamma}] \\ &= \frac{1}{3}\left(\sum_{(\alpha,\beta,\gamma)} \nabla_\alpha R^\lambda_{\gamma\beta n}\right) - \frac{1}{6}[\nabla_\beta(R^\lambda_{\alpha\gamma n} - 2R^\lambda_{\gamma\alpha n}) + \nabla_\gamma(R^\lambda_{\alpha\beta n} + R^\lambda_{\beta\alpha n}) \\ &+ \nabla_\alpha(R^\lambda_{\beta\gamma n} - 2R^\lambda_{\gamma\beta n}) - \nabla_n(R^\lambda_{\alpha\beta\gamma} + R^\lambda_{\beta\alpha\gamma})] \\ &= \frac{1}{2}\left(\nabla_\alpha R^\lambda_{\gamma\beta n} + \nabla_\beta R^\lambda_{\gamma\alpha n}\right) + \frac{1}{6}\nabla_n\left(R^\lambda_{\alpha\beta\gamma} + R^\lambda_{\beta\alpha\gamma}\right) \\ &+ \frac{1}{6}\left[\nabla_\alpha\left(R^\lambda_{\gamma\beta n} - R^\lambda_{\beta\gamma n}\right) - 2\nabla_\beta\left(R^\lambda_{\alpha\gamma n} - R^\lambda_{\gamma\alpha n}\right) + \nabla_\gamma\left(R^\lambda_{\beta\alpha n} - R^\lambda_{\alpha\beta n}\right)\right] \ .\end{aligned}$$

By the first Bianchi identity, we have: $\nabla_\alpha\left(R^\lambda_{\gamma\beta n} - R^\lambda_{\beta\gamma n}\right) = \nabla_\alpha R^\lambda_{n\beta\gamma}$ and similarly for the two other differences occuring in the last brackets. Combining this with the second Bianchi identity now routinely yields formula (32). Lemma 2 is proved.

Finally, we will require yet another set of identities, involving *third* derivatives of the Christoffel symbols. Unlike the preceding ones, they will hold only *modulo* addition of a linear combination of terms, each of which being a component (in the Fermi chart, on the axis) of one of the following three tensors:

$$(\mathrm{Riem} - \mathrm{Cur}_1), \nabla(\mathrm{Riem} - \mathrm{Cur}_1) \equiv \nabla \mathrm{Riem}, \nabla^2(\mathrm{Riem} - \mathrm{Cur}_1) \equiv \nabla^2 \mathrm{Riem},$$

and the absolute value of each coefficient of the linear combination being bounded above by some constant under control (thus independent of $(m_0, v_0) \in \mathrm{NoCut}$ and of $\delta > 0$). In the sequel, an equality modulo the addition of such a linear combination will be denoted by "$\simeq$". Recalling (10), if two scalars A and B satisfy $\mathrm{A} \simeq \mathrm{B}$, there exists a constant under control c such that $|\mathrm{A} - \mathrm{B}| \leq c\,\delta$. This is exactly the type of inequality allowed for proving Theorem 2. The proof of the next lemma will illustrate the use of these notations.

Lemma 3 *The following 'equalities' hold on the axis:*

$$\partial_{\alpha\beta\gamma}\Gamma^n_{nn} \simeq 0 \ , \ \partial_{\alpha\beta\gamma}\Gamma^\lambda_{nn} \simeq -\frac{4}{3}\sum_{(\alpha,\beta,\gamma)} \delta_{\lambda\alpha}\delta_{\gamma\beta} \ .$$

Proof. Brut calculation yields on the axis:

$$\nabla_{\alpha\beta} R^n_{n\gamma n} = \partial_{\alpha\beta} R^n_{n\gamma n} + \left(\partial_\alpha \Gamma^n_{\beta\lambda}\right) R^\lambda_{n\gamma n} - \left(\partial_\alpha \Gamma^\lambda_{\beta n}\right) R^n_{\lambda\gamma n} \ .$$

Using (29) and the identity:

$$\begin{aligned} \mathrm{Riem} \otimes \mathrm{Riem} \ &\equiv \ \mathrm{Riem} \otimes (\mathrm{Riem} - \mathrm{Cur}_1) \\ &+ \ (\mathrm{Riem} - \mathrm{Cur}_1) \otimes \mathrm{Cur}_1 + \mathrm{Cur}_1 \otimes \mathrm{Cur}_1 \ , \end{aligned} \tag{37}$$

we readily infer: $\partial_{\alpha\beta} R^n_{n\gamma n} \simeq 0$. Calculation again yields:

$$\begin{aligned} \partial_{\alpha\beta} R^n_{n\gamma n} = \partial_{\alpha\beta\gamma}\Gamma^n_{nn} - \partial_n\left(\partial_{\alpha\beta}\Gamma^n_{n\gamma}\right) \ &+ \ \partial_\alpha \Gamma^j_{nn}\partial_\beta \Gamma^n_{\gamma j} + \partial_\alpha \Gamma^n_{\gamma j}\partial_\beta \Gamma^j_{nn} \\ &- \ \partial_\alpha \Gamma^j_{\gamma n}\partial_\beta \Gamma^n_{nj} - \partial_\alpha \Gamma^n_{nj}\partial_\beta \Gamma^j_{\gamma n} \ . \end{aligned}$$

Now we use (31), and (29) combined with (37), to obtain:

$$\partial_{\alpha\beta} R^n_{n\gamma n} \simeq \partial_{\alpha\beta\gamma}\Gamma^n_{nn} \ ,$$

thus proving the first formula.
For the second formula, we first observe the 'equality':

$$\nabla_{\alpha\beta}(\mathrm{Riem} - \mathrm{Cur}_1)^\lambda_{n\gamma n} \simeq \partial_{\alpha\beta}(\mathrm{Riem} - \mathrm{Cur}_1)^\lambda_{n\gamma n}$$

and compute each term of the right-hand side. Using (28) and $\mathrm{Riem} \simeq \mathrm{Cur}_1$, we find:

$$\partial_{\alpha\beta}(\mathrm{Cur}_1)^\lambda_{n\gamma n} = \delta^\lambda_\gamma \partial_{\alpha\beta} g_{nn} \simeq -2\delta_{\lambda\gamma}\delta_{\alpha\beta} \tag{38}$$

Brut calculation yields for the other term:

$$\begin{aligned} \partial_{\alpha\beta} R^\lambda_{n\gamma n} = \partial_{\alpha\beta\gamma}\Gamma^\lambda_{nn} - \partial_n\left(\partial_{\alpha\beta}\Gamma^\lambda_{n\gamma}\right) \ &+ \ \partial_\alpha \Gamma^j_{nn}\partial_\beta \Gamma^\lambda_{\gamma j} + \partial_\alpha \Gamma^\lambda_{\gamma j}\partial_\beta \Gamma^j_{nn} \\ &- \ \partial_\alpha \Gamma^j_{\gamma n}\partial_\beta \Gamma^\lambda_{nj} - \partial_\alpha \Gamma^\lambda_{nj}\partial_\beta \Gamma^j_{\gamma n} \ . \end{aligned} \tag{39}$$

Combining (29) with (37), we find for the last two terms of the right-hand side:

$$-(\partial_\alpha \Gamma^j_{\gamma n}\partial_\beta \Gamma^\lambda_{nj} + \partial_\alpha \Gamma^\lambda_{nj}\partial_\beta \Gamma^j_{\gamma n}) \simeq (\delta_{\lambda\alpha}\delta_{\gamma\beta} + \delta_{\lambda\beta}\delta_{\gamma\alpha}) \ .$$

To cope with the two preceding terms, we apply to (39) the circular sum over (α, β, γ); by symmetry, the last 'equality' and (38) yield, recalling (32):

$$\partial_{\alpha\beta\gamma}\Gamma^\lambda_{nn} \simeq -\frac{4}{3}\sum_{(\alpha,\beta,\gamma)} (\delta_{\lambda\alpha}\delta_{\gamma\beta} + \frac{1}{4}\partial_\alpha \Gamma^j_{nn}\partial_\beta \Gamma^\lambda_{\gamma j} + \frac{1}{4}\partial_\alpha \Gamma^\lambda_{\gamma j}\partial_\beta \Gamma^j_{nn}) \ .$$

To treat the last two terms, noting the 'equality' $\partial_\alpha \Gamma^j_{nn} \simeq \delta_{\alpha j}$ which follows from (29), we are lead to study the circular sum: $\sum_{(\alpha,\beta,\gamma)} (\partial_\beta \Gamma^\lambda_{\gamma\alpha} + \partial_\alpha \Gamma^\lambda_{\gamma\beta})$. Brut calculation on the axis yields:

$$\partial_\beta \Gamma^\lambda_{\gamma\alpha} + \partial_\alpha \Gamma^\lambda_{\gamma\beta} = -\frac{1}{2}(\partial_{\beta\mu} g_{\alpha\gamma} + \partial_{\alpha\mu} g_{\beta\gamma} - \partial_{\alpha\beta} g_{\mu\gamma}) \ ,$$

so, by (33), circular summation cancels the last term of the latter right-hand side and we readily find:

$$\sum_{(\alpha,\beta,\gamma)} (\partial_\beta \Gamma^\lambda_{\gamma\alpha} + \partial_\alpha \Gamma^\lambda_{\gamma\beta}) = -\sum_{(\alpha,\beta,\gamma)} \partial_{\alpha\mu} g_{\beta\gamma}$$

which vanishes by (34). Lemma 3 is proved.

2.2 Hessian of the squared distance from a point

Let us fix a point $m_0 \in M_n$ and a normal chart $x = \left(x^1, \dots, x^n\right)$ centered at m_0. For an arbitrary geodesic segment $[m, \exp_m(v)]$ contained in the domain of our chart, with $(m, v) \in \mathrm{NoCut}$, it will be convenient to stick to a normalized 'time' parameter $t \in [0, 1]$. We will set $\left(v^1, \dots, v^n\right)$ for the fiber coordinates of the chart of TM_n naturally associated to the chart x, and:

$$X = X(x^1, \dots, x^n, v^1, \dots, v^n, t) := x \circ [\exp_m(tv)] =: \left(X^1, \dots, X^n\right) \ ,$$

thus with $v = v^i \partial_i$. To compute the local expression $A(x, v)$ of the quadratic form defined by (9) at $x = 0$, we start from the well-known identity [26, p.156]:

$$p_2 \equiv \exp_{p_1}\left[-\operatorname{grad}_{p_1} c(p_1, p_2)\right] \ , \tag{40}$$

valid whenever $(p_1, p_2) \in M_n^2$ are not cut-points of each other. Taking the points p_a's lying in the domain of our chart and setting $x_a = x(p_a)$, we differentiate (40) with respect to the coordinates x_1^j's at $x_1 = 0$, getting for $X(x_1, v, t)$ at $x_1 = 0, t = 1$ and at $v = v^i\partial_i$ given by $\exp_{m_0}(v) = p_2$, the following identity:

$$0 \equiv \frac{\partial X^i}{\partial x_1^j}(0, v, 1) - \sum_{k=1}^n \frac{\partial X^i}{\partial v^k}(0, v, 1)\frac{\partial^2 c}{\partial x_1^j \partial x_1^k}[0, X(0, v, 1)] \ .$$

We may thus write, in matrix form (and dropping the subscript of x_1):

$$A(0, v) \equiv \left[\frac{\partial X}{\partial v}(0, v, 1)\right]^{-1}\left[\frac{\partial X}{\partial x}(0, v, 1)\right] \ . \tag{41}$$

This is the fundamental formula to be used for the calculation at m_0 of the c-curvature (8). It leads us to compute the matrix coefficients of $\left[\frac{\partial X}{\partial x}(x,v,t)\right]$ and $\left[\frac{\partial X}{\partial v}(x,v,t)\right]$ in the next section, then the first and second partial derivatives of $\left[\frac{\partial X}{\partial x}(0,v,t)\right]$ and $\left[\frac{\partial X}{\partial v}(0,v,t)\right]$ with respect to the fiber variable v respectively in sections 2.4 and 2.5.

2.3 First derivatives of geodesic motion

Preliminary bounds.
In this section and the next two, we will proceed stepwise, deriving first a bound under control on the g-norms of the x and v derivatives of X and $\dot{X} = \frac{dX}{dt}$ under study calculated at $(0, v_0, t)$. Then we will compare these derivatives with the ones which would occur in the constant curvature 1 case and prove that the g-norms of the differences between the two are $\simeq 0$.
The strategy to get a bound under control on derivatives of X and $\dot{X}$ with respect to the initial conditions (x, v), calculated at $(0, v_0, t)$, goes as follows. Any such derivative of X, denote it by $J(t)$, will satisfy constant initial conditions and solve the Jacobi equation along the geodesic γ_0 (which reads $t \mapsto X(0, v_0, t)$), possibly in non-homogeneous form, which we write here (with standard notations specified below):

$$\ddot{J} + \mathrm{Riem}(J, \dot{\gamma}_0)\dot{\gamma}_0 = P \ ,$$

where the right-hand side P will be a polynomial expression in the (previously kept under control) lower order derivatives of X and $\dot{X}$, with only local Riemannian invariants as coefficients. Granted this, the estimation scheme is standard; let us sketch it here once for all.

Standard estimation scheme. Transform the Jacobi equation into a first order system (S) bearing on the auxiliary variable:

$$K := \begin{pmatrix} J \\ \dot{J} \end{pmatrix}$$

and compute $\dfrac{d}{dt}$ of the squared norm $|K|^2 = |J|^2 + |\dot{J}|^2$. Using the system (S) combined with the triangle and the Schwarz inequalities, get a constant under control C such that:

$$\frac{d|K|^2}{dt} \leq C\left(1 + |K|^2\right),$$

and conclude: $1 + |K|^2(t) \leq \left[1 + |K|^2(0)\right] e^{Ct}$.

First derivatives calculations
Henceforth, we fix $(m_0, v_0) \in \mathrm{NoCut}$ with $v_0 \neq 0$ (unless otherwise specified) and an associated Fermi chart. The n-tuple $X = X(x, v, t)$ is the solution of the following Cauchy problem:

$$\ddot{X}^i + \Gamma^i_{jk}(X)\dot{X}^j\dot{X}^k = 0,\ X^i(0) = x^i,\ \dot{X}^i(0) = v^i \ . \tag{42}$$

dots standing for time derivatives. By differentiating that problem with respect to the parameters x^a or v^a, we get the following equation satisfied by J_a (equal to either $\partial_{x^a} X$ or $\partial_{v^a} X$):

$$\ddot{J}^i_a + \partial_l \Gamma^i_{jk}(X) \dot{X}^j \dot{X}^k J^l_a + 2\Gamma^i_{jk}(X) \dot{X}^j \dot{J}^k_a = 0 \ , \tag{43}$$

with the correspondingly differentiated initial conditions, namely either:

$$\partial_{x^a} X^i(0) = \delta^i_a, \ \partial_{x^a} \dot{X}^i(0) = 0, \tag{44}$$

or:

$$\partial_{v^a} X^i(0) = 0, \ \partial_{v^a} \dot{X}^i(0) = \delta^i_a \ . \tag{45}$$

On the axis, setting for short $X_0(t) := X(0, v_0, t)$ and recalling (29), equation (43) becomes:

$$\ddot{J}^i_a + |v_0|^2 R^i_{n\alpha n}(X_0) J^\alpha_a = 0 \ , \tag{46}$$

or else, in coordinate-free form, setting $\gamma_0(t) := \exp_{m_0}(tv_0)$ (so $X_0 \equiv x \circ \gamma_0$)):

$$\ddot{J}_a + \mathrm{Riem}(J_a, \dot{\gamma_0})\dot{\gamma_0} = 0 \ ;$$

we recognize the Jacobi equation[1]. For later use, let us record a basic fact (*cf. supra*) from second order differential equations theory:

Lemma 4 *There exists a constant $c_1 > 0$ under control such that, for each $t \in [0,1]$, the following g-norms:*

$$|\partial_x X(0, v_0, t)|, \ |\partial_x \dot{X}(0, v_0, t)|, \ |\partial_v X(0, v_0, t)|, \ |\partial_v \dot{X}(0, v_0, t)|,$$

are all bounded above by c_1; here, the g-norm of $\partial_x X(x, v, t)$ is defined by:

$$|\partial_x X(x, v, t)|^2 = g_{ij}[X(x, v, t)] g^{kl}[X(x, v, t)] \frac{\partial X^i}{\partial x^k}(x, v, t) \frac{\partial X^j}{\partial x^l}(x, v, t)$$

and similarly for $|\partial_x \dot{X}(x, v, t)|, |\partial_v X(x, v, t)|, |\partial_v \dot{X}(x, v, t)|$.

Let us rewrite the Jacobi equation in the perturbative form:

$$\ddot{J}_a + \mathrm{Cur}_1(J_a, \dot{\gamma_0})\dot{\gamma_0} = (\mathrm{Cur}_1 - \mathrm{Riem})(J_a, \dot{\gamma_0})\dot{\gamma_0}$$

(where it is understood, here and below, that the tensors Cur_1 and Riem are considered at γ_0) which will enable us to use assumption (10). The preceding equation reads $\ddot{J}^n_a = 0$ and:

$$\forall \alpha < n, \ \ddot{J}^\alpha_a + |v_0|^2 J^\alpha_a = |v_0|^2 (\mathrm{Cur}_1 - \mathrm{Riem})^\alpha_{n\gamma n} J^\gamma_a \ . \tag{47}$$

We will require the notation $\overline{\partial_x X_0}(t)$ (resp. $\overline{\partial_v X_0}(t)$) for the solution $\bar{J}_a$ of the, so to say, *unperturbed* equation

$$\ddot{\bar{J}}_a + \mathrm{Cur}_1(\bar{J}_a, \dot{X}_0)\dot{X}_0 = 0$$

satisfying the same initial conditions (44) (resp. (45)) as $\partial_x X$ (resp. $\partial_v X$).

Lemma 5 *In the Fermi chart, on the axis, the first derivatives of the geodesic motion with respect to the initial conditions satisfy, for each $t \in [0,1]$, the following g-norm bounds:*

$$\max\left(|\partial_x X(0, v_0, t) - \overline{\partial_x X_0}(t)|, |\partial_v X(0, v_0, t) - \overline{\partial_v X_0}(t)|\right) \leq 2c_1\delta \ .$$

[1] in [18, p.307], equation (43) is improperly called so

Proof. From (47), we readily find for $\partial_x X(0,v_0,t)$ (resp. $\partial_v X(0,v_0,t)$) and for $\overline{\partial_x X_0}(t)$ (resp. $\overline{\partial_v X_0}(t)$) the *same* axis components, namely:

(48) $\partial_{x^n} X_0^n = 1,\ \partial_{v^n} X_0^n = t,\ \partial_{x^\alpha} X_0^n = \partial_{v^\alpha} X_0^n = \partial_{x^n} X_0^\alpha = \partial_{v^n} X_0^\alpha = 0\ .$

We thus focus on the J_β^α components. We require a lemma (easily verified):

Lemma 6 (representation formula) *Given a function $t \mapsto \varphi(t)$ and a real number $\omega_0 \neq 0$, set:*

$$\psi = \ddot{\varphi} + \omega_0^2 \varphi,\ \ \lambda = \varphi(0),\ \ \mu = \dot{\varphi}(0).$$

The following identity holds:

$$\varphi(t) = \lambda \cos(\omega_0 t) + \mu \frac{\sin(\omega_0 t)}{\omega_0} + \sin(\omega_0 t) \int_0^t \frac{1}{\sin^2(\omega_0 \tau)} \left[\int_0^\tau \sin(\omega_0 \theta) \psi(\theta) d\theta \right] d\tau.$$

Applying Lemma 6 to $\varphi = J_\beta^\alpha$ (with $\omega_0 = |v_0|$), equation (47) implies

$$\partial_{x^\beta} X_0^\alpha(t) = \delta_\beta^\alpha \cos(|v_0| t) + \mathcal{E}_{x^\beta}^\alpha(t) \equiv \left(\overline{\partial_x X_0}\right)_\beta^\alpha(t) + \mathcal{E}_{x^\beta}^\alpha(t)$$

with the x-correction term given by:

$$\mathcal{E}_{x^\beta}^\alpha(t) = |v_0|^2 \sin(|v_0| t) \int_0^t \frac{1}{\sin^2(|v_0|\tau)} \left[\int_0^\tau \sin(|v_0|\theta) (\mathrm{Cur}_1 - \mathrm{Riem})_{n\gamma n}^\alpha \partial_{x^\beta} X_0^\gamma(\theta)\ d\theta \right] d\tau\ ,$$

and

$$\partial_{v^\beta} X_0^\alpha(t) = \delta_\beta^\alpha \frac{\sin(|v_0| t)}{|v_0|} + \mathcal{E}_{v^\beta}^\alpha(t) \equiv \left(\overline{\partial_v X_0}\right)_\beta^\alpha(t) + \mathcal{E}_{v^\beta}^\alpha(t)$$

with the v-correction term given by:

$$\mathcal{E}_{v^\beta}^\alpha(t) = |v_0|^2 \sin(|v_0| t) \int_0^t \frac{1}{\sin^2(|v_0|\tau)} \left[\int_0^\tau \sin(|v_0|\theta) (\mathrm{Cur}_1 - \mathrm{Riem})_{n\gamma n}^\alpha \partial_{v^\beta} X_0^\gamma(\theta)\ d\theta \right] d\tau\ .$$

Using (10), Schwarz inequality and Lemma 4, we infer for the Euclidean norm of both x and v error $(n-1)\times(n-1)$ matrices $\mathcal{E} = \left[\mathcal{E}_\beta^\alpha(t)\right]$ the upper bound $|\mathcal{E}| \leq \delta c_1\ \Im_{|v_0|}(t)$ with:

$$\Im_{\omega_0}(t) := \omega_0^2\ \sin(\omega_0 t) \int_0^t \frac{1}{\sin^2(\omega_0 \tau)} \left[\int_0^\tau \sin(\omega_0 \theta) d\theta \right] d\tau\ .$$

Now Lemma 5 follows from the following technical one (left as an exercise):

Lemma 7 *For $(\omega_0, t) \in [0,\pi] \times [0,1]$, the following equality holds:*

$$\Im_{\omega_0}(t) = 1 - \cos(\omega_0 t)\ .$$

Remark 4 For later use, dealing with $|\partial_v X(0,v_0,t) - \overline{\partial_v X_0}(t)|$, let us note that the constant c_1 of Lemma 5 may be taken equal to $\sqrt{n-1}$. Indeed, on the one hand, the proof of Lemma 5 combined with Lemma 7 and Schwarz inequality provides the g-norms inequality:

$$|\partial_v X(0,v_0,t) - \overline{\partial_v X_0}(t)| \leq 2\delta \max_{\theta\in[0,1]} \sqrt{\sum_{\beta,\gamma} \left[\partial_{v^\beta} X_0^\gamma(\theta)\right]^2}\ .$$

On the other hand, for each $\beta \in \{0, \ldots, n-1\}$, using (5) and the strict inequality $|v_0| < \pi$, we can apply the Rauch comparison theorem [13, p.29] [10, p.215] to the Jacobi field $\partial_{v^\beta} X_0(t)$ along the axis and readily infer from it the upper bound:

$$\forall \theta \in [0,1], \ \sum_\gamma \left[\partial_{v^\beta} X_0^\gamma(\theta)\right]^2 \leq \sum_\gamma \left[\overline{\partial_{v^\beta} X_0^\gamma(\theta)}\right]^2 = \left(\frac{\sin|v_0|\theta}{|v_0|}\right)^2 \leq 1.$$

The claim follows by summing over $\beta < n$ the resulting inequality, taking the square root of each side and the maximum over $\theta \in [0,1]$.

We will require a similar result for the time derivative of $\partial_x X$ and $\partial_v X$, namely:

Lemma 8 *In the Fermi chart, on the axis, the first derivatives of the time derivative of the geodesic motion with respect to the initial conditions satisfy:*

$$\max\left(|\partial_x \dot{X}(0,v_0,t) - \dot{\overline{\partial_x X_0}}(t)|, |\partial_v \dot{X}(0,v_0,t) - \dot{\overline{\partial_v X_0}}(t)|\right) \leq c_1' \delta$$

for some constant under control $c_1' > 0$ independent of $t \in [0,1]$.

Proof. All axis components of the differences under study vanish, so let us focus on the sole components $\mathcal{E}_\beta^\alpha$ (the subscript β standing for either x^β or v^β) which satisfy, recalling (47):

$$\ddot{\mathcal{E}}_\beta^\alpha + |v_0|^2 \mathcal{E}_\beta^\alpha \simeq 0$$

with null initial conditions. The latter yields the representation:

$$\dot{\mathcal{E}}_\beta^\alpha(t) = \int_0^t \ddot{\mathcal{E}}_\beta^\alpha(\tau) d\tau \ ,$$

hence the former, combined with the triangle inequality and Lemma 5, implies:

$$|\dot{\mathcal{E}}| \leq c_1' \delta$$

with a constant c_1' under control, as required.

2.4 Second derivatives of geodesic motion

Differentiating with respect to the parameter v^b (component of the initial velocity in the Fermi chart) the Cauchy problems (43)-(44) or (43)-(45), and sticking to the notation J_a^i used there, yields the following equation satisfied at $X = X(0,v,t)$ by $J_{ab} \equiv J_{ba}$ (an admittedly loose but typographically convenient abbreviation, in which the subscript a will be the sole one to stand for either x^a or v^a, other subscripts $b, c, \ldots$ standing only for $v^b, v^c, \ldots$; it will enable us, in the next subsection, to write (for short) sums involving the J_{ab}'s as circular sums) with J_{ab} equal either to $\partial^2_{x^a v^b} X = \partial^2_{x^a v^b} X(0,v,t)$ or to $\partial^2_{v^a v^b} X = \partial^2_{v^a v^b} X(0,v,t)$:

$$\begin{aligned}(49) \qquad \ddot{J}^i_{ab} \ &+ \ \left(\partial_l \Gamma^i_{jk}\right) \dot{X}^j \dot{X}^k J^l_{ab} + 2\Gamma^i_{jk} \dot{X}^j \dot{J}^k_{ab} = \\ &- \ \left(\partial_{lm} \Gamma^i_{jk}\right) \dot{X}^j \dot{X}^k J^l_a J^m_b - 2\left(\partial_l \Gamma^i_{jk}\right) \dot{X}^j \left(\dot{J}^k_b J^l_a + \dot{J}^k_a J^l_b\right) \\ &- \ 2\Gamma^i_{jk} \dot{J}^j_b \dot{J}^k_a \ .\end{aligned}$$

and (in either case) the *null* initial conditions:

$$J_{ab}(0) = 0,\ \dot{J}_{ab}(0) = 0\ . \tag{50}$$

Along the axis, recalling (29), equation (49) reads:

$$\ddot{J}^i_{ab} + |v_0|^2 R^i_{n\gamma n}(X_0) J^\gamma_{ab} = -|v_0|^2 \left(\partial_{lm}\Gamma^i_{nn}\right) J^l_a J^m_b - 2|v_0| R^i_{k\beta n}\left(\dot{J}^k_b J^\beta_a + \dot{J}^k_a J^\beta_b\right).$$

Using (29)(30) and Lemma 4 to treat the latter right-hand side, we may once again record a standard result of second order differential equations theory, namely:

Lemma 9 *There exists a constant $c_2 > 0$ under control such that, for each $t \in [0,1]$, the following g-norms:*

$$|\partial^2_{xv} X(0, v_0, t)|,\ |\partial^2_{xv}\dot{X}(0, v_0, t)|,\ |\partial^2_{vv} X(0, v_0, t)|,\ |\partial^2_{vv}\dot{X}(0, v_0, t)|,$$

are all bounded above by c_2.

Let us rewrite the above equation in perturbative form, namely:

$$\begin{aligned} \ddot{J}^i_{ab} + |v_0|^2 \delta^i_\gamma J^\gamma_{ab} &= |v_0|^2 \left(\mathrm{Cur}_1 - \mathrm{Riem}\right)^i_{n\gamma n} J^\gamma_{ab} \\ &- |v_0|^2 \left(\partial_{lm}\Gamma^i_{nn}\right) J^l_a J^m_b - 2|v_0| R^i_{k\beta n}\left(\dot{J}^k_b J^\beta_a + \dot{J}^k_a J^\beta_b\right)\ . \end{aligned} \tag{51}$$

Using (29)(30) to treat the right-hand side, we find:

$$\begin{aligned} \ddot{J}^i_{ab} + |v_0|^2 \delta^i_\gamma J^\gamma_{ab} &\simeq -2|v_0| \left(\mathrm{Cur}_1\right)^i_{k\beta n}\left(\dot{J}^k_b J^\beta_a + \dot{J}^k_a J^\beta_b\right) \\ &= -2|v_0| \left(\delta_{i\beta}\delta_{kn} - \delta_{in}\delta_{k\beta}\right)\left(\dot{J}^k_b J^\beta_a + \dot{J}^k_a J^\beta_b\right)\ , \end{aligned}$$

or else, if $i = \alpha$:

$$\ddot{J}^\alpha_{ab} + |v_0|^2 J^\alpha_{ab} \simeq -2|v_0| \left(\dot{J}^n_b J^\alpha_a + \dot{J}^n_a J^\alpha_b\right)\ , \tag{52}$$

while if $i = n$:

$$\ddot{J}^n_{ab} \simeq 2|v_0| \sum_{\beta=1}^{n-1}\left(\dot{J}^\beta_b J^\beta_a + \dot{J}^\beta_a J^\beta_b\right)\ . \tag{53}$$

Recalling (48), if $a = b = n$, we infer at once:

$$\ddot{J}^i_{nn} + |v_0|^2 \delta^i_\gamma J^\gamma_{nn} \simeq 0\ ; \tag{54}$$

moreover, if a or b is equal to n, we get from (53), say with $b = n$:

$$\ddot{J}^n_{an} \simeq 0\ . \tag{55}$$

Let us treat equation (53) in the remaining cases for (a, b). If $a = \lambda \neq b = \mu$, the combination of (53) with Lemmas 4, 5 and 8, implies the existence of a constant $c_{21} > 0$ under control such that:

$$\max_{\lambda\neq\mu, t\in[0,1]} \left|\dot{J}^n_{\lambda\mu}\right|(0, v_0, t) \leq c_{21}\delta\ . \tag{56}$$

Finally, if $a = b = \lambda$, sticking to the auxiliary notation $\bar{J}^i_a$ of the preceding subsection, we write:

$$\dot{J}^\beta_\lambda J^\beta_\lambda = \dot{\bar{J}}^\beta_\lambda \bar{J}^\beta_\lambda + \dot{\bar{J}}^\beta_\lambda \left(J^\beta_\lambda - \bar{J}^\beta_\lambda \right) + \left(\dot{J}^\beta_\lambda - \dot{\bar{J}}^\beta_\lambda \right) J^\beta_\lambda$$

and, recalling Lemmas 5 and 8, we obtain the existence of a constant under control $c_{22} > 0$ such that either (if $a = b = v^\lambda$):

$$\max_{\lambda, t \in [0,1]} \left| \ddot{J}^n_{\lambda\lambda} - 4|v_0| \sum_{\beta=1}^{n-1} \dot{\bar{J}}^\beta_\lambda \bar{J}^\beta_\lambda \right| \leq c_{22}\ \delta\ , \tag{57}$$

or (if $a = x^\lambda$, thus $b = v^\lambda$):

$$\max_{\lambda, t \in [0,1]} \left| \ddot{J}^n_{\lambda\lambda} - 2|v_0| \sum_{\beta=1}^{n-1} \left(\dot{\bar{J}}^\beta_{x^\lambda} \bar{J}^\beta_{v^\lambda} + \bar{J}^\beta_{x^\lambda} \dot{\bar{J}}^\beta_{v^\lambda} \right) \right| \leq c_{22}\ \delta\ , \tag{58}$$

Let us turn to equation (52) in case a or b differs from n; we must distinguish cases. If both differ from n, we infer from (48) that the quantity $\left(\dot{J}^n_b J^\alpha_a + \dot{J}^n_a J^\alpha_b \right)$ vanishes; so there exists a constant under control $c_{23} > 0$ such that:

$$\max_{\lambda, \mu, t \in [0,1]} \left| \ddot{J}^\alpha_{\lambda\mu} + |v_0|^2 J^\alpha_{\lambda\mu} \right| \leq c_{23}\ \delta. \tag{59}$$

If a stands for x^n (b thus differing from n), we infer similarly the vanishing of $\left(\dot{J}^n_b J^\alpha_a + \dot{J}^n_a J^\alpha_b \right)$ hence the existence of a constant under control $c_{24} > 0$ such that:

$$\max_{\lambda, t \in [0,1]} \left| \ddot{J}^\alpha_{x^n\lambda} + |v_0|^2 J^\alpha_{x^n\lambda} \right| \leq c_{24}\ \delta. \tag{60}$$

If a or b stands for v^n, still using (48) and taking (say) $a = v^n, b = \lambda$, we find: $\left(\dot{J}^n_b J^\alpha_a + \dot{J}^n_a J^\alpha_b \right) = J^\alpha_\lambda$. If $\lambda \neq \alpha$, Lemma 5 implies the existence of a constant under control $c_{25} > 0$ such that:

$$\max_{\lambda \neq \alpha, t \in [0,1]} \left| \ddot{J}^\alpha_{v^n\lambda} + |v_0|^2 J^\alpha_{v^n\lambda} \right| \leq c_{25}\ \delta, \tag{61}$$

while if $\lambda = \alpha$, it implies the existence of a constant under control $c_{26} > 0$ such that:

$$\max_{\alpha, t \in [0,1]} \left| \ddot{J}^\alpha_{v^n\alpha} + |v_0|^2 J^\alpha_{v^n\alpha} + 2|v_0| \bar{J}^\alpha_\alpha \right| \leq c_{26}\ \delta. \tag{62}$$

At this stage, sticking to the intermediate notations $\bar{J}^i_a$ of the preceding section, let us introduce the solutions $\overline{\partial^2_{xv} X_0}$ and $\overline{\partial^2_{vv} X_0}$ along the axis of the unperturbed equation:

$$\ddot{\bar{J}}^i_{ab} + |v_0|^2 \delta^i_\gamma \bar{J}^\gamma_{ab} = -2|v_0| \left(\delta_{i\beta}\delta_{kn} - \delta_{in}\delta_{k\beta} \right) \left(\bar{J}^\beta_a \dot{\bar{J}}^k_b + \bar{J}^\beta_b \dot{\bar{J}}^k_a \right) \tag{63}$$

still with null initial conditions.

Lemma 10 *There exists a constant $c_{27} > 0$ under control such that, for each $t \in [0,1]$, the following g-norms:*

$$\left| \partial^2_{xv} X(0, v_0, t) - \overline{\partial^2_{xv} X_0}(t) \right|, \quad \left| \partial^2_{vv} X(0, v_0, t) - \overline{\partial^2_{vv} X_0}(t) \right|,$$

are bounded above by $c_{27}\delta$.

Proof. Setting $\mathcal{E}^i_{ab}(t)$ for the components of the difference under study and combining (54)(55)(56)(57)(58)(59)(60)(61)(62), we find that $\mathcal{E}^i_{ab}$ satisfies:

$$\ddot{\mathcal{E}}^i_{ab} + |v_0|^2 \delta^i_\gamma \mathcal{E}^\gamma_{ab} \simeq 0$$

with null initial conditions. Applying Lemma 6 to $\mathcal{E}^i_{ab}$, as done above, yields the desired upper bound on its g-norm.
Besides, since $\mathcal{E}^i_{ab}$ solves the preceding Cauchy problem, we may argue as in the proof of Lemma 8 and immediately obtain:

Lemma 11 *There exists a constant under control $c_2' > 0$ such that, for each $t \in [0,1]$, the following g-norms:*

$$\left|\partial^2_{xv}\dot{X}(0,v_0,t) - \dot{\overline{\partial^2_{xv}X_0}}(t)\right|, \ \left|\partial^2_{vv}\dot{X}(0,v_0,t) - \dot{\overline{\partial^2_{vv}X_0}}(t)\right|,$$

are bounded above by $c_2'\delta$.

2.5 Third derivatives of geodesic motion

Differentiating with respect to the initial velocity component parameter v^c the Cauchy problems (49)-(50) yields on the axis the following equation for $J_{abc}(t)$ equal to, either $\partial^3_{x^a v^b v^c}X(0,v_0,t)$ or to $\partial^3_{v^a v^b v^c}X(0,v_0,t)$, after use of (29):

$$\begin{aligned}
(64) \qquad \ddot{J}^i_{abc} + |v_0|^2 R^i_{n\gamma n} J^\gamma_{abc} &= -|v_0|^2 \left(\partial_{lmp}\Gamma^i_{nn}\right) J^l_a J^m_b J^p_c \\
&- |v_0|^2 \left(\partial_{lm}\Gamma^i_{nn}\right) \sum_{(a,b,c)} J^l_{ab} J^m_c \\
&- 2|v_0| \left(\partial_{lm}\Gamma^i_{nk}\right) \sum_{(a,b,c)} \dot{J}^k_a J^l_b J^m_c \\
&- 2|v_0| R^i_{k\beta n} \sum_{(a,b,c)} \left(\dot{J}^k_a J^\beta_{bc} + \dot{J}^k_{ab} J^\beta_c\right) \\
&- 2\left(\partial_\beta \Gamma^i_{jk}\right) \sum_{(a,b,c)} \dot{J}^j_a \dot{J}^k_b J^\beta_c \ ,
\end{aligned}$$

still with null initial conditions. Here, we will require the full strength of Lemmas 2 and 3 to check the intrinsic character of the right hand-side coefficients of the J's and $\dot{J}$'s. Granted this is done, recalling Lemmas 4 and 9, we may already record a standard result of second order ODE theory, namely:

Lemma 12 *There exists a constant $c_3 > 0$ under control such that, for each $t \in [0,1]$, the following g-norms:*

$$\left|\partial^3_{xvv}X(0,v_0,t)\right|, \left|\partial^3_{vvv}X(0,v_0,t)\right|,$$

are bounded above by c_3.

To proceed further with Equation (64), let us distinguish cases.

First case: $i = n$. The equation reads:

$$\ddot{J}^n_{abc} = \mathrm{I}_{abc} + \mathrm{II}_{abc} + \mathrm{III}_{abc} + \mathrm{IV}_{abc} + \mathrm{V}_{abc}$$

where:

$$\mathrm{I}_{abc} := -|v_0|^2\,(\partial_{lmp}\Gamma^n_{nn})\,J^l_a J^m_b J^p_c$$

is $\simeq 0$ due to a combination of (29)(30) and the first formula of Lemma 3, with Lemma 4; then:

$$\mathrm{II}_{abc} := -|v_0|^2\,(\partial_{lm}\Gamma^n_{nn}) \sum_{(a,b,c)} J^l_{ab} J^m_c$$

is $\simeq 0$ due to (29)(30) combined with Lemmas 4 and 9; besides:

$$\mathrm{III}_{abc} := -2|v_0|\,(\partial_{lm}\Gamma^n_{nk}) \sum_{(a,b,c)} \dot{J}^k_a J^l_b J^m_c$$

is $\simeq 0$ due to (29)(30)(31) combined with Lemma 4; furthermore:

$$\mathrm{IV}_{abc} := 2|v_0| R^n_{\beta n\gamma} \sum_{(a,b,c)} \left(\dot{J}^\beta_a J^\gamma_{bc} + \dot{J}^\beta_{ab} J^\gamma_c\right)$$

becomes, using Lemmas 4 and 9:

$$\mathrm{IV}_{abc} \simeq 2|v_0| \sum_{\beta<n} \sum_{(a,b,c)} \left(\dot{J}^\beta_a J^\beta_{bc} + \dot{J}^\beta_{ab} J^\beta_c\right) \ ,$$

or else, in terms of the above spherical quantities $\bar{J}_a, \bar{J}_{bc}$, after use of the finite differences trick combined with Lemmas 5, 8, 10, 11:

$$\mathrm{IV}_{abc} \simeq 2|v_0| \sum_{\beta<n} \sum_{(a,b,c)} \left(\dot{\bar{J}}^\beta_a \bar{J}^\beta_{bc} + \dot{\bar{J}}^\beta_{ab} \bar{J}^\beta_c\right) \ ;$$

last:

$$\mathrm{V}_{abc} := -2\left(\partial_\beta \Gamma^n_{jk}\right) \sum_{(a,b,c)} \dot{J}^j_a \dot{J}^k_b J^\beta_c \ ,$$

splits into a sum over $j<n$ and $k<n$, which is by (29) equal to:

$$\frac{2}{3}\left(R^n_{\lambda\mu\beta} + R^n_{\mu\lambda\beta}\right) \sum_{(a,b,c)} \dot{J}^\lambda_a \dot{J}^\mu_b J^\beta_c$$

and so, using Lemma 4, which is $\simeq 0$, and a sum for j or k equal to n which, by (29), reads:

$$2R^n_{\gamma n\beta} \sum_{(a,b,c)} \left(\dot{J}^n_a \dot{J}^\gamma_b J^\beta_c + \dot{J}^\gamma_a \dot{J}^n_b J^\beta_c\right) \ ,$$

hence, by Lemma 4:

$$\mathrm{V}_{abc} \simeq 2 \sum_{\beta<n} \sum_{(a,b,c)} \left(\dot{J}^n_a \dot{J}^\beta_b J^\beta_c + \dot{J}^\beta_a \dot{J}^n_b J^\beta_c\right) \ ,$$

and, finally, combining the finite differences trick with Lemmas 5 and 8:

$$\mathrm{V}_{abc} \simeq 2 \sum_{\beta<n} \sum_{(a,b,c)} \left(\dot{\bar{J}}^n_a \dot{\bar{J}}^\beta_b \bar{J}^\beta_c + \dot{\bar{J}}^\beta_a \dot{\bar{J}}^n_b \bar{J}^\beta_c\right) \ .$$

Altogether, Equation (64) with $i = n$ thus yields:

$$\ddot{J}^n_{abc} \simeq 2 \sum_{\beta<n} \sum_{(a,b,c)} \left[|v_0| \left(\dot{\bar{J}}^\beta_a \bar{J}^\beta_{bc} + \dot{\bar{J}}^\beta_{ab} \bar{J}^\beta_c \right) + \left(\dot{\bar{J}}^n_a \dot{\bar{J}}^\beta_b \bar{J}^\beta_c + \dot{\bar{J}}^\beta_a \dot{\bar{J}}^n_b \bar{J}^\beta_c \right) \right] .$$

Let us set $\bar{J}^n_{abc}(t)$ for the solution of the unperturbed equation:

$$\ddot{\bar{J}}^n_{abc} = 2 \sum_{\beta<n} \sum_{(a,b,c)} \left[|v_0| \left(\dot{\bar{J}}^\beta_a \bar{J}^\beta_{bc} + \dot{\bar{J}}^\beta_{ab} \bar{J}^\beta_c \right) + \left(\dot{\bar{J}}^n_a \dot{\bar{J}}^\beta_b \bar{J}^\beta_c + \dot{\bar{J}}^\beta_a \dot{\bar{J}}^n_b \bar{J}^\beta_c \right) \right]$$

with null initial conditions, and $\mathcal{E}^n_{abc}$ for the difference $J^n_{abc} - \bar{J}^n_{abc}$ which satisfies:

$$\ddot{\mathcal{E}}^n_{abc} \simeq 0, \ \mathcal{E}^n_{abc}(0) = \dot{\mathcal{E}}^n_{abc}(0) = 0 .$$

The latter implies the existence of a constant under control $c > 0$ such that:

$$\forall t \in [0,1], \ |\mathcal{E}^n_{abc}(t)| \leq c\delta . \tag{65}$$

Second case: $i = \rho < n$. In that case, Equation (64) written in perturbative form reads as follows:

$$\ddot{J}^\rho_{abc} + |v_0|^2 J^\rho_{abc} = \mathrm{I}^\rho_{abc} + \mathrm{II}^\rho_{abc} + \mathrm{III}^\rho_{abc} + \mathrm{IV}^\rho_{abc} + \mathrm{V}^\rho_{abc} + \mathrm{VI}^\rho_{abc} ,$$

with:

$$\mathrm{I}^\rho_{abc} := |v_0|^2 \left(\mathrm{Cur}_1 - \mathrm{Riem}\right)^\rho_{n\gamma n} J^\gamma_{abc} ,$$

$$\mathrm{II}^\rho_{abc} := -\frac{1}{3} |v_0|^2 \left(\partial_{lmp} \Gamma^\rho_{nn}\right) \sum_{(a,b,c)} J^l_a J^m_b J^p_c ,$$

$$\mathrm{III}^\rho_{abc} := -|v_0|^2 \left(\partial_{lm} \Gamma^\rho_{nn}\right) \sum_{(a,b,c)} J^l_{ab} J^m_c ,$$

$$\mathrm{IV}^\rho_{abc} := -2|v_0| \left(\partial_{lm} \Gamma^\rho_{nk}\right) \sum_{(a,b,c)} \dot{J}^k_a J^l_b J^m_c ,$$

$$\mathrm{V}^\rho_{abc} := -2|v_0| R^\rho_{k\beta n} \sum_{(a,b,c)} \left(\dot{J}^k_a J^\beta_{bc} + \dot{J}^k_{ab} J^\beta_c \right) ,$$

$$\mathrm{VI}^\rho_{abc} := -2 \left(\partial_\beta \Gamma^\rho_{jk} \right) \sum_{(a,b,c)} \dot{J}^j_a \dot{J}^k_b J^\beta_c .$$

Deferring the treatment of I^ρ_{abc}, let us proceed with the other terms. Each summand of II^ρ_{abc} with l, m, or p equal to n, is $\simeq 0$ by (29) and (30) combined with Lemma 4; using the latter and the second formula of Lemma 3, we infer:

$$\mathrm{II}^\rho_{abc} \simeq \frac{4}{9} |v_0|^2 \sum_{(\lambda,\mu,\nu)} \sum_{(a,b,c)} \delta_{\lambda\rho} \delta_{\mu\nu} J^\lambda_a J^\mu_b J^\nu_c = \frac{4}{3} |v_0|^2 \sum_{(a,b,c)} J^\rho_a \sum_{\mu<n} J^\mu_b J^\mu_c .$$

After use of the finite differences trick combined with Lemmas 4 and 5, we thus obtain:

$$\mathrm{II}^\rho_{abc} \simeq \overline{\mathrm{II}}^\rho_{abc} := \frac{4}{3} |v_0|^2 \sum_{(a,b,c)} \bar{J}^\rho_a \sum_{\mu<n} \bar{J}^\mu_b \bar{J}^\mu_c .$$

By (29) and (30) combined with Lemmas 4 and 9, we have $\mathrm{III}^{\rho}_{abc} \simeq 0$. Each summand of IV^{ρ}_{abc} with l, m, or k equal to n, is $\simeq 0$ by (29) and (30) combined with Lemma 4; moreover, by (32) combined with Lemma 4, the remaining sum bearing on $(l, m, k) = (\lambda, \mu, \nu)$ is $\simeq 0$ as well. Next, Lemmas 4 and 9 yield:

$$\mathrm{V}^{\rho}_{abc} \simeq -2|v_0| \sum_{(a,b,c)} \left(\dot{J}^n_a J^{\rho}_{bc} + \dot{J}^n_{ab} J^{\rho}_c \right);$$

combining the finite differences trick with Lemmas 5, 8, 10 and 11, we thus get:

$$\mathrm{V}^{\rho}_{abc} \simeq \overline{\mathrm{V}}^{\rho}_{abc} := -2|v_0| \sum_{(a,b,c)} \left(\dot{\bar{J}}^n_a \bar{J}^{\rho}_{bc} + \dot{\bar{J}}^n_{ab} \bar{J}^{\rho}_c \right).$$

Finally, let us write $\mathrm{VI}^{\rho}_{abc} = (\text{VI-1})^{\rho}_{abc} + (\text{VI-2})^{\rho}_{abc} + (\text{VI-3})^{\rho}_{abc}$ with:

$$(\text{VI-1})^{\rho}_{abc} := -2 \left(\partial_\beta \Gamma^{\rho}_{nn}\right) \sum_{(a,b,c)} \dot{J}^n_a \dot{J}^n_b J^{\beta}_c \ ,$$

$$(\text{VI-2})^{\rho}_{abc} := -2 \left(\partial_\beta \Gamma^{\rho}_{\lambda n}\right) \sum_{(a,b,c)} \left(\dot{J}^{\lambda}_a \dot{J}^n_b J^{\beta}_c + \dot{J}^n_a \dot{J}^{\lambda}_b J^{\beta}_c \right) \ ,$$

$$(\text{VI-3})^{\rho}_{abc} := -2 \left(\partial_\beta \Gamma^{\rho}_{\lambda\mu}\right) \sum_{(a,b,c)} \dot{J}^{\lambda}_a \dot{J}^{\mu}_b J^{\beta}_c \ .$$

From (29), we have:

$$(\text{VI-1})^{\rho}_{abc} = -2R^{\rho}_{n\beta n} \sum_{(a,b,c)} \dot{J}^n_a \dot{J}^n_b J^{\beta}_c \ ,$$

$$(\text{VI-2})^{\rho}_{abc} = -2R^{\rho}_{\lambda\beta n} \sum_{(a,b,c)} \left(\dot{J}^{\lambda}_a \dot{J}^n_b J^{\beta}_c + \dot{J}^n_a \dot{J}^{\lambda}_b J^{\beta}_c \right),$$

$$(\text{VI-3})^{\rho}_{abc} = -\frac{2}{3} \left(R^{\rho}_{\lambda\beta\mu} + R^{\rho}_{\mu\beta\lambda} \right) \sum_{(a,b,c)} \dot{J}^{\lambda}_a \dot{J}^{\mu}_b J^{\beta}_c \ .$$

Using Lemma 4, we get $(\text{VI-1})^{\rho}_{abc} \simeq -2 \sum_{(a,b,c)} \dot{J}^n_a \dot{J}^n_b J^{\rho}_c$, $(\text{VI-2})^{\rho}_{abc} \simeq 0$ and:

$$\begin{aligned} (\text{VI-3})^{\rho}_{abc} &\simeq -\frac{2}{3} \left(2\delta^{\rho}_{\beta}\delta_{\lambda\mu} - \delta^{\rho}_{\mu}\delta_{\lambda\beta} - \delta^{\rho}_{\lambda}\delta_{\mu\beta} \right) \sum_{(a,b,c)} \dot{J}^{\lambda}_a \dot{J}^{\mu}_b J^{\beta}_c \\ &= \frac{2}{3} \sum_{\mu<n} \sum_{(a,b,c)} \left[\left(\dot{J}^{\rho}_a \dot{J}^{\mu}_b + \dot{J}^{\mu}_a \dot{J}^{\rho}_b \right) J^{\mu}_c - 2\dot{J}^{\mu}_a \dot{J}^{\mu}_b J^{\rho}_c \right]. \end{aligned}$$

Combining the finite differences trick with Lemmas 5 and 8, we find:

$$(\text{VI-1})^{\rho}_{abc} \simeq \overline{(\text{VI-1})}^{\rho}_{abc} := -2 \sum_{(a,b,c)} \dot{\bar{J}}^n_a \dot{\bar{J}}^n_b \bar{J}^{\rho}_c \ ,$$

$$(\text{VI-3})^{\rho}_{abc} \simeq \overline{(\text{VI-3})}^{\rho}_{abc} := \frac{2}{3} \sum_{\mu<n} \sum_{(a,b,c)} \left[\left(\dot{\bar{J}}^{\rho}_a \dot{\bar{J}}^{\mu}_b + \dot{\bar{J}}^{\mu}_a \dot{\bar{J}}^{\rho}_b \right) \bar{J}^{\mu}_c - 2\dot{\bar{J}}^{\mu}_a \dot{\bar{J}}^{\mu}_b \bar{J}^{\rho}_c \right].$$

Back to the, yet untreated, right-hand term I^ρ_{abc}, we may now use Lemma 12 which implies: $\mathrm{I}^\rho_{abc} \simeq 0$.
Let us set $\bar{J}^\rho_{abc}(t)$ for the solution of the unperturbed equation:

$$\ddot{\bar{J}}^\rho_{abc} + |v_0|^2 \bar{J}^\rho_{abc} = \overline{\mathrm{II}}^\rho_{abc} + \overline{\mathrm{V}}^\rho_{abc} + \overline{(\text{VI-1})}^\rho_{abc} + \overline{(\text{VI-3})}^\rho_{abc}$$

with null initial conditions, and $\mathcal{E}^\rho_{abc}$ for the difference $J^\rho_{abc} - \bar{J}^\rho_{abc}$. By construction, $\mathcal{E}^\rho_{abc}$ satisfies:

$$\ddot{\mathcal{E}}^\rho_{abc} \simeq 0, \mathcal{E}^\rho_{abc}(0) = \dot{\mathcal{E}}^\rho_{abc}(0) = 0,$$

hence there exists a constant under control $c' > 0$ such that:

$$\forall t \in [0,1], \ |\mathcal{E}^\rho_{abc}(t)| \le c'\delta \ . \tag{66}$$

Setting $\overline{\partial^3_{xvv} X_0}(t) = \bar{J}^i_{x^a v^b v^c}(t) dx^a \otimes dv^b \otimes dv^c \otimes \frac{\partial}{\partial x^i}$ and similarly for $\overline{\partial^3_{vvv} X_0}(t)$, we can express our results (65)(66) by the following statement:

Lemma 13 *There exists a constant under control $c_3 > 0$ such that, for each $t \in [0,1]$, the following g-norms:*

$$\left|\partial^3_{xvv} X(0, v_0, t) - \overline{\partial^3_{xvv} X_0}(t)\right|, \ \left|\partial^3_{vvv} X(0, v_0, t) - \overline{\partial^3_{vvv} X_0}(t)\right|,$$

are bounded above by $c_3\delta$.

2.6 Perturbative c-curvature calculation

We are now in position to complete the proof of Theorem 2. Given a fixed couple of orthogonal unit vectors $\xi \perp \nu$ in $T_{m_0}M_n$, let us go back to the defining expression (8) of the c-curvature $\mathcal{C}(m_0, v_0)(\xi, \nu)$ and compute it in a normal chart at m_0, starting from the local formula (41). Set, for short:

$$J^i_{v^k} = \frac{\partial X^i}{\partial v^k}(0, v, 1), \ J^i_{x^k} = \frac{\partial X^i}{\partial x^k}(0, v, 1),$$

and (Y^i_k) for the $n \times n$ matrix inverse of $(J^i_{v^k})$. Near $v = v_0$, the local matrix field $v \mapsto (Y^i_k)$ satisfies:

$$Y^l_i J^i_{v^k} = \delta^l_k \text{ ,hence in turn } dY^l_j = -Y^l_i Y^k_j dJ^i_{v^k} \ . \tag{67}$$

From (9)(41), setting $\xi_k dx^k := g(\xi, .)$, we thus start from the expression:

$$A(v)(\xi) := A(m_0, v)(\xi) = Y^k_i J^i_{x^j} \xi^j \xi_k \ ,$$

apply twice to it the (vertical, flat) derivative $\partial_\nu = \nu^m \dfrac{\partial .}{\partial v^m}$, then let $v = v_0$. Using repeatedly (67), we routinely obtain (with obvious notations to abbreviate second and third derivatives of $J^i = X^i(x, v, t)$ at $(0, v_0, 1)$, as well) the general local expression of the c-curvature in any normal chart at m_0, namely:

$$\begin{aligned} \mathcal{C}(m_0, v_0)(\xi, \nu) &= -\nu^l \nu^m \partial^2_{v^l v^m} A(v)(\xi)|_{v=v_0} \\ &= [2 Y^p_i Y^k_q J^i_{x^j v^l} J^q_{v^m v^p} \\ &- (Y^r_i Y^p_s Y^k_q + Y^p_i Y^r_q Y^k_s) J^i_{x^j} J^q_{v^l v^p} J^s_{v^m v^r} \\ &- Y^k_i J^i_{x^j v^l v^m} + Y^p_i Y^k_q J^i_{x^j} J^q_{v^l v^m v^p}] \ \nu^l \nu^m \xi^j \xi_k \ . \end{aligned} \tag{68}$$

Remark 5 As a simple application of that formula, let us calculate the expression of the c-curvature in the special case $v_0 = 0$. The geodesic $\gamma_0(t) = \exp_{m_0}(tv_0)$ is then constant, equal to m_0. Using a Riemannian normal chart at m_0, Eq. (43) (resp. Eq. (49)) read along $X(0,0,t) \equiv 0$ and supplemented by the initial conditions (44) or (45) (resp. (50)) yields immediately:

$$\partial_{x^a} X^i(t) = \delta^i_a,\ \partial_{v^a} X^i(t) = t\delta^i_a,\ \partial^2_{ab} X^i((t) \equiv 0.$$

In particular, we thus have: $Y^a_i = \delta^a_i$. Moreover, differentiating Eq. (49) with respect to v^c, taking null initial conditions and using the preceding equalities, we get at once:

$$\partial^3_{x^a v^b v^c} X^i(t) = -t^2 \partial_a \Gamma^i_{bc}(0) \equiv -\frac{1}{3}\left(R^i_{bac} + R^i_{cab}\right)(0),$$

$$\partial^3_{v^a v^b v^c} X^i(t) = -\frac{t^3}{3} \sum_{(a,b,c)} \partial_a \Gamma^i_{bc}(0) \equiv 0\ ,$$

where the former identity goes back to Riemann [11, Eq. (22), p.244] and the latter vanishing is thus due to the first Bianchi identity. Plugging all these values into Formula (68), we obtain:

$$\mathcal{C}(m_0,0)(\xi,\nu) = -J^k_{x^j v^l v^m} \nu^l \nu^m \xi^j \xi_k = \frac{2}{3} R^k_{ljm}(0) \nu^l \nu^m \xi^j \xi_k\ ;$$

in other words, indeed [31, 28], we find $\mathcal{C}(m_0,0)(\xi,\nu)$ equal to the 2/3-rd of the sectional curvature of (M_n, g) at the 2-plane defined by m_0 and (ξ,ν).

Using the local barred quantities introduced in the preceding three sections, henceforth understood taken at $t = 1$ (unless otherwise specified), and setting $(\bar{Y}^i_k)$ for the inverse matrix of $(\bar{J}^i_{v^k})$, one can express similarly the spherical c-curvature $\overline{\mathcal{C}}(m_0,v_0)(\xi,\nu)$. Doing so, and using the finite differences trick in a systematic way, we find for the c-curvatures difference the following expression:

$$\mathcal{C}(m_0,v_0)(\xi,\nu) - \overline{\mathcal{C}}(m_0,v_0)(\xi,\nu) = \tag{69}$$

$$\begin{gathered}
\{2(Y^p_i - \bar{Y}^p_i)\bar{Y}^k_q \bar{J}^i_{x^j v^l} \bar{J}^q_{v^m v^p} + 2Y^p_i (Y^k_q - \bar{Y}^k_q) \bar{J}^i_{x^j v^l} \bar{J}^q_{v^m v^p} \\
+\ 2Y^p_i Y^k_q (J^i_{x^j v^l} - \bar{J}^i_{x^j v^l}) \bar{J}^q_{v^m v^p} + 2Y^p_i Y^k_q J^i_{x^j v^l} (J^q_{v^m v^p} - \bar{J}^q_{v^m v^p}) \\
-[(Y^r_i - \bar{Y}^r_i)\bar{Y}^p_s \bar{Y}^k_q + Y^r_i (Y^p_s - \bar{Y}^p_s)\bar{Y}^k_q + Y^r_i Y^p_s (Y^k_q - \bar{Y}^k_q) \\
+(Y^p_i - \bar{Y}^p_i)\bar{Y}^r_q \bar{Y}^k_s +\ Y^p_i (Y^r_q - \bar{Y}^r_q)\bar{Y}^k_s + Y^p_i Y^r_q (Y^k_s - \bar{Y}^k_s)] \bar{J}^i_{x^j} \bar{J}^q_{v^l v^p} \bar{J}^s_{v^m v^r} \\
-(Y^r_i Y^p_s Y^k_q + Y^p_i Y^r_q Y^k_s)\ [(J^i_{x^j} - \bar{J}^i_{x^j}) \bar{J}^q_{v^l v^p} \bar{J}^s_{v^m v^r} + J^i_{x^j} (J^q_{v^l v^p} - \bar{J}^q_{v^l v^p}) \bar{J}^s_{v^m v^r} \\
+J^i_{x^j} J^q_{v^l v^p} (J^s_{v^m v^r} - \bar{J}^s_{v^m v^r})] - (Y^k_i - \bar{Y}^k_i) \bar{J}^i_{x^j v^l v^m} - Y^k_i (J^i_{x^j v^l v^m} - \bar{J}^i_{x^j v^l v^m}) \\
+(Y^p_i - \bar{Y}^p_i)\bar{Y}^k_q \bar{J}^i_{x^j} \bar{J}^q_{v^l v^m v^p} + Y^p_i (Y^k_q - \bar{Y}^k_q) \bar{J}^i_{x^j} \bar{J}^q_{v^l v^m v^p} \\
+Y^p_i Y^k_q (J^i_{x^j} - \bar{J}^i_{x^j}) \bar{J}^q_{v^l v^m v^p} + Y^p_i Y^k_q J^i_{x^j} (J^q_{v^l v^m v^p} - \bar{J}^q_{v^l v^m v^p})\}\ \nu^l \nu^m \xi^j \xi_k\ .
\end{gathered}$$

It is important, here, that we record (in connection with the constant C_2 of Theorem 2) the particular structure of the right-hand side of Equation (69): apart from the unit-vectors ξ, ν of course, it involves only $Y, \bar{Y}$, the J's and the $\bar{J}$'s; it does it in a polynomial way; moreover, each summand contains exactly one of the differences $(Y-\bar{Y}),(J-\bar{J})$. With the view of proving the estimate (12), let us evaluate a difference like $(Y_i^l-\bar{Y}_i^l)$ in terms of the differences $(J^j_{v^k}-\bar{J}^j_{v^k})$. To do so, we first write:

$$J^i_{v^k}=\bar{J}^i_{v^j}[\delta^j_k-\bar{Y}^j_l(\bar{J}^l_{v^k}-J^l_{v^k})]$$

and, setting provisionally $\mu^l_j:=\bar{Y}^l_k(\bar{J}^k_{v^j}-J^k_{v^j})$, we infer the formal expansion:

$$Y^l_i=\left(\delta^l_j+\mu^l_j+\mu^{l_1}_j\mu^l_{l_1}+\sum_{N=2}^{\infty}\mu^{l_1}_j\mu^{l_2}_{l_1}\dots\mu^{l_N}_{l_{N-1}}\mu^l_{l_N}\right)\bar{Y}^j_i\ .$$

Assuming $v_0\neq 0$ and using a Fermi chart associated to (m_0,v_0), we have

$$\bar{J}^\alpha_{v^a}=\delta^\alpha_a\frac{\sin(|v_0|t)}{|v_0|}$$

as well-known (*cf. e.g.* [18]), hence:

$$\bar{Y}^\alpha_a=\delta^\alpha_a\frac{|v_0|}{\sin|v_0|},\ \bar{Y}^n_a=\delta^n_a\ .$$

Moreover, recalling (48), the sole differences $(J^l_{v^k}-\bar{J}^l_{v^k})$ to take in account will be those for k and l smaller than n. Recalling Lemma 5 and Remark 4, we set $\mathcal{D}^\alpha_\beta:=\bar{J}^\alpha_\beta-J^\alpha_\beta$, thus with the g-norm bound $|\mathcal{D}|\leq 2\delta\sqrt{n-1}$; writing $\mu^l_j=\delta^l_\alpha\delta^\beta_j\frac{|v_0|}{\sin|v_0|}\mathcal{D}^\alpha_\beta$, we infer from the above expansion that $(Y^l_i-\bar{Y}^l_i)$ is formally equal to:

$$\delta^l_\alpha\delta^\beta_i\left(\frac{|v_0|}{\sin|v_0|}\right)^2\mathcal{D}^\alpha_\gamma\left[\delta^\gamma_\beta+\frac{|v_0|}{\sin|v_0|}\mathcal{D}^\gamma_\beta+\sum_{N=2}^{\infty}\left(\frac{|v_0|}{\sin|v_0|}\right)^N\mathcal{D}^\gamma_{\gamma_1}\mathcal{D}^{\gamma_1}_{\gamma_2}\dots\mathcal{D}^{\gamma_{N-1}}_\beta\right].$$

The condition (11) of Theorem 2 implies: $\frac{|v_0|}{\sin|v_0|}|\mathcal{D}|\leq\frac{1}{2}$, which ensures the uniform convergence of the latter expansion and yields the g-norm upper bound:

$$|Y-\bar{Y}|\leq 4\sqrt{n-1}\left(\frac{|v_0|}{\sin|v_0|}\right)^2\delta\ . \tag{70}$$

The latter, combined with the triangle inequality and (11), provides the upper bound:

$$\sqrt{\sum_{\alpha,\beta}\left(Y^\alpha_\beta\right)^2}\leq 2\sqrt{n-1}\frac{|v_0|}{\sin|v_0|}\ . \tag{71}$$

By a lengthy but routine inspection of each of its summand, we can now estimate the right-hand side of (69), using repeatedly the triangle and Schwarz inequalities combined with (70)(71) (and $Y^n_a=\delta^n_a, Y^a_n=\delta^a_n$), the inequality $\frac{|v_0|}{\sin|v_0|}\geq 1$ and Lemmas 4, 5, 9, 10, 12, 13, and obtain the existence of a constant $C_2\geq 1$ under control such that:

$$\left|\mathcal{C}(m_0,v_0)(\xi,\nu)-\overline{\mathcal{C}}(m_0,v_0)(\xi,\nu)\right|\leq C_2\left(\frac{|v_0|}{\sin|v_0|}\right)^4\delta\ .$$

Last, we note that the function $\theta \mapsto \frac{\theta}{\sin\theta}$ is increasing from 1 to ∞ on $[0,\pi)$, where it satisfies the following (easily verified) inequality:

$$\frac{\theta}{\sin\theta} \le \frac{\pi}{\pi-\theta} \ .$$

The latter yields for $|v_0| = (1-\eta_0)\pi$, with $\eta_0 \in (0,1)$, the upper bound:

$$\frac{|v_0|}{\sin|v_0|} \le \frac{1}{\eta_0} \ ;$$

so the proof of Theorem 2 is complete.
In order to test the sharpness of the resulting bound (12), let us exhibit a summand of (69) which is $O\left(\frac{\delta}{\eta_0^4}\right)$ as η_0 goes to 0. Among the terms of the sum:

$$-Y_i^r Y_s^p (Y_q^k - \bar{Y}_q^k) \bar{J}^i_{x^j} \bar{J}^q_{v^l v^p} \bar{J}^s_{v^m v^r} \nu^l \nu^m \xi^j \xi_k \ ,$$

fixing $\alpha \in \{1,\dots,n-1\}$, take those with:

$$l = m = n, \ p = q = r = k = s = \alpha$$

(the latter equalities imply $i = j =: \beta < n$), which reads:

$$-\sum_{\beta<n} \xi_\alpha \xi^\beta \left(\nu^n\right)^2 Y_\beta^\alpha Y_\alpha^\alpha \left(Y_\alpha^\alpha - \bar{Y}_\alpha^\alpha\right) \bar{J}^\beta_{x^\beta} \left(\bar{J}^\alpha_{v^n v^\alpha}\right)^2 =: T_\alpha \ .$$

At $t=1$, we have $\bar{J}^\beta_{x^\beta} = \cos|v_0|$ for each $\beta < n$, and:

$$\bar{J}^\alpha_{v^n v^\alpha} = \frac{1}{|v_0|^2}(|v_0|\cos|v_0| - \sin|v_0|),$$

as readily checked. So there exists a constant $c \ge 1$ (independent of $(m_0, v_0), \delta$ and n) such that:

$$|T_\alpha| \le c\,|Y_\alpha^\alpha|\left|Y_\alpha^\alpha - \bar{Y}_\alpha^\alpha\right| \sum_{\beta<n} \left|Y_\beta^\alpha\right|$$

hence also, by (70)(71) and the expression of $\bar{Y}_\alpha^\alpha$ (*cf. supra*), such that:

$$|T_\alpha| \le 16(n-1)^2 c \left(\frac{|v_0|}{\sin|v_0|}\right)^4 \delta \ .$$

A bound on $|T_\alpha|$ of order $O\left(\frac{\delta}{\eta_0^4}\right)$ thus, indeed, occurs as $\eta_0 \downarrow 0$.

Remark 6 In Theorem 2, we may take the constant C_2 such that, for some integer k, the quantity $C_2 n^{-k/2}$ remains bounded as $n \to \infty$. The existence of such an integer k follows by a careful inspection of our estimates of Sections 2.3 through 2.6, provided the initial standard estimation scheme used for Lemmas 4, 9 and 12, is replaced by the improved *ad hoc* scheme described below. Granted it, using extensively the triangle and Schwarz inequalities (for the norm and scalar product g_{m_0}) combined with (10) and (11), each estimate derived in the aforementioned sections turns out, indeed, polynomial in the ultimate variables:

$$\max_{t\in[0,1]} |\partial_x \overline{X_0}(t)| = \max_{t\in[0,1]} |\partial_v \overline{X_0}(t)| = \sqrt{n}, \text{ and } |\mathrm{Cur}_1| = \sqrt{2n(n-1)},$$

with universal constants as coefficients (*N.B.* the bounds for the barred quantities are obtained from the others by letting $\mathrm{Riem} = \mathrm{Cur}_1$ and $\delta = 0$).
***Ad hoc* estimation scheme.** Rewrite the non-homogeneous Jacobi equation under study in the form:

$$\ddot{J} + \mathrm{Cur}_1(J, \dot\gamma_0)\dot\gamma_0 = (\mathrm{Cur}_1 - \mathrm{Riem})(J, \dot\gamma_0)\dot\gamma_0 + P$$

and use the representation device of Lemmas 6 and 7 for its solution, combined with condition (10) and the Schwarz and triangle inequalities, to get:

$$\max_{t\in[0,1]} |J(t)| \le |J(0)| + |\dot J(0)| + 2\delta \max_{t\in[0,1]} |J(t)| + 2 \max_{t\in[0,1]} |P(t)|.$$

Recalling (11), conclude:

$$\max_{t\in[0,1]} |J(t)| \le 2\left(|J(0)| + |\dot J(0)| + 2 \max_{t\in[0,1]} |P(t)| \right).$$

Here, either $|J(0)|$ or $|\dot J(0)|$ is equal to $\sqrt{n}$, the other one vanishing, in case we deal with first derivatives of $X(x, v, t)$ at $(0, v_0, t)$, or $|J(0)| = |\dot J(0)| = 0$ in case we deal with higher order derivatives.
Derive the estimate on $|\dot J(t)|$ from the equation, by writing:

$$\dot J(t) = \dot J(0) + \int_0^t \ddot J(\tau)d\tau = \dot J(0) + \int_0^t [P(\tau) - \mathrm{Riem}(J, \dot\gamma_0)\dot\gamma_0]d\tau$$

and by using the preceding estimate on $|J(\tau)|$ (combined again with the Schwarz and triangle inequality).

A Spherical c-curvature calculations

For completeness, we provide here the proof of inequality (17) and thus redo formally some of Loeper's calculations [31]. Fixing $(m_0, v_0) \in \mathrm{NoCut}$ with $v_0 \neq 0$ and a couple (ξ, ν) of orthogonal unit vectors in $T_{m_0}M_n$, let us compute

$$\overline{\mathcal{C}}(m_0, v_0)(\xi, \nu) = -Dd[v \mapsto \overline{A}(m_0, v)(\xi)]|_{v=v_0}(\nu, \nu)$$

where D stands for the canonical flat connection of $T_{m_0}M_n$ and $\overline{A}(m_0, v)(\xi)$ is given by $\overline{A}(m_0, v)(\xi) = 1 - \varphi(v)h(v)$ with $\varphi(v) = \Phi(|v|) := 1 - |v| \cot |v|$ and $h(v) := 1 - <\xi, \frac{v}{|v|}>^2$ (setting $<.,.> := g_{m_0}(.,.)$ for short). We readily get for $\overline{\mathcal{C}}(m_0, v_0)(\xi, \nu)$ the expression:

$$h(v_0)Dd\varphi(v_0)(\nu, \nu) + 2d\varphi(v_0)(\nu)dh(v_0)(\nu) + \varphi(v_0)Ddh(v_0)(\nu, \nu)$$

or else:

$$\begin{aligned} &h(v_0)\left\{\Phi'(|v_0|)[Dd|v|(\nu, \nu)]_{v=v_0} + \Phi''(|v_0|)[d|v|(\nu)]^2_{v=v_0}\right\} \\ &+2\Phi'(|v_0|)[d|v|(\nu)]_{v=v_0} dh(v_0)(\nu) + \varphi(v_0)Ddh(v_0)(\nu, \nu). \end{aligned}$$

Using the auxiliary formulas:

$$d|v|(\nu) = <\nu, \frac{v}{|v|}> \quad , \quad Dd|v|(\nu, \nu) = \frac{1}{|v|}\left(1 - <\nu, \frac{v}{|v|}>^2\right) ,$$

$$d<\xi, \frac{v}{|v|}>(\nu) = - \frac{1}{|v|} <\xi, \frac{v}{|v|}> <\nu, \frac{v}{|v|}> ,$$

$$\Phi'(r) = \frac{r - \sin r \cos r}{\sin^2 r} \quad , \quad \Phi''(r) = \frac{2}{\sin^2 r}\Phi(r) ,$$

and setting for short: $r = |v_0|, \overline{v_0} = \frac{v_0}{r}$, we find $\overline{\mathcal{C}}(m_0, v_0)(\xi, \nu)$ equal to:

$$\left[1 - <\xi, \overline{v_0}>^2\right] \left\{ \frac{r - \cos r \sin r}{r \sin^2 r} \left[1 - <\nu, \overline{v_0}>^2\right] + \frac{2(\sin r - r \cos r)}{\sin^3 r} <\nu, \overline{v_0}>^2 \right\}$$

$$+<\xi, \overline{v_0}>^2 \left[\frac{4(r - \cos r \sin r)}{r \sin^2 r} <\nu, \overline{v_0}>^2 + \frac{2(\sin r - r \cos r)}{r^2 \sin r} \left(1 - 4<\nu, \overline{v_0}>^2\right) \right].$$

Applying the easily established inequalities:

$$\forall t \in [0, \pi],\ \sin t - t \cos t \geq \frac{t^3}{\pi^2},\ t - \sin t \cos t \geq \frac{t^3}{\pi^2},$$

and setting:

$$\begin{aligned} P(x, y, z) &:= z[z(1 - x)(1 - y + 2yz) + 2x(1 - y)]\ , \\ \Psi(t) &:= 2t^2 - 3 \sin^2 t + t \cos t \sin t\ , \end{aligned}$$

we infer the lower bound:

$$\overline{\mathcal{C}}(m_0, v_0)(\xi, \nu) \geq \frac{1}{\pi^2} P\left(<\xi, \overline{v_0}>^2, <\nu, \overline{v_0}>^2, \frac{r}{\sin r}\right) + \frac{2<\xi, \overline{v_0}>^2 <\nu, \overline{v_0}>^2}{r^2 \sin^2 r} \Psi(r).$$

A lengthy but routine check (differentiating six times the function ζ defined on $[0, 2\pi]$ by $\zeta(t) := \Psi(t/2)$) shows that the function Ψ is non-negative on $[0, \pi]$. So

$$\overline{\mathcal{C}}(m_0, v_0)(\xi, \nu) \geq \frac{1}{\pi^2} P(x, y, z)$$

with

$$x := <\xi, \overline{v_0}>^2, y := <\nu, \overline{v_0}>^2, z := \frac{|v_0|}{\sin |v_0|},$$

satisfying:

(72) $$x \geq 0,\ y \geq 0,\ x + y \leq 1,\ z \geq 1\ .$$

From the latter inequality, we have $P(x, y, z) \geq z\ Q(x, y)$ with:

$$Q(x, y) := 1 + x + y - 3xy\ .$$

Using the arithmetic–geometric inequality, we get

$$Q(x, y) \geq 1 + (x + y) \left[1 - \frac{3}{4}(x + y)\right]$$

hence, by (72), we have $Q(x, y) \geq 1$ and:

$$\overline{\mathcal{C}}(m_0, v_0)(\xi, \nu) \geq \frac{1}{\pi^2} \frac{|v_0|}{\sin |v_0|}.$$

Finally, on the one hand, we have $\frac{|v_0|}{\sin |v_0|} \geq 1$, on the other hand, since $|v_0| = (1 - \eta_0)\pi$ and $\sin |v_0| = \sin \eta_0 \pi \leq \eta_0 \pi$, we also have $\frac{|v_0|}{\sin |v_0|} \geq \frac{1 - \eta_0}{\eta_0}$. Altogether, we obtain the lower bound (17) as claimed.

B The Ma–Trudinger–Wang estimate

The interior C^2 estimate carried out in [33, Theorem 4.1] requires preliminary bounds, notably on the cost-function c up to its fourth partial derivatives (in some local charts). We need to adapt it to our manifold context in order to keep track of an intrinsic control on all auxiliary quantities.

B.1 Expressing the optimal transport equation

Fix $(m_0, V_0) \in \mathrm{NoCut}$ and let x (resp. y) be a chart of M_n at m_0 (resp. at $p_0 = \exp_{m_0}(V_0)$) with $x(m_0) = 0$. Set (x, v) for the natural chart of TM_n associated to x, with $(x, v)(m_0, V_0) = (0, v_0) \in \mathbb{R}^n \times \mathbb{R}^n$, and for $(m, V) \in TM_n$ close to (m_0, V_0), set

$$E(x, v) := y[\exp_m(V)]$$

where $x = x(m)$ and $V = v^i \partial_{x^i}$. Consider the real function Φ defined near (m_0, V_0) in TM_n by:

$$\Phi(m, V) = \frac{\sqrt{|g|}(x)}{\sqrt{|g|}(E(x, v)) \det\left(\frac{\partial E}{\partial v}(x, v)\right)} ,$$

where the same symbol $\sqrt{|g|}$ abusively denotes the Riemannian density in either charts x or y; so, for instance: $\mathrm{dVol}(m) = \sqrt{|g|}(x)dx^1 \dots dx^n$. One can routinely check that the function Φ is independent of the choice of the charts x and y; as such, it is globally defined on NoCut. We set:

$$\forall (m, V) \in \mathrm{NoCut}, \forall t \in [0, 1],\ B_t(m, V) := \frac{\rho_0(m)}{\rho_t(\exp_m V)} \Phi(m, V)$$

(where the function ρ_t is the one defined in the statement of Theorem 1). Now, Equation (1) globally reads as follows [16]:

$$\forall m \in M_n, \frac{\det \mathrm{Hess}^{(c)} u_t}{\det g}(m) = B_t(m, \mathrm{grad}_m u_t) . \tag{73}$$

In order to fit with the local setting of [33, Theorem 4.1], we will require another expression of it, a local one, attached to a couple of charts x and y as above. Fixing henceforth $t \in \mathcal{T}$, we set:

$$\psi(x, v) := \log\left[\det g(m)\ B_t(m, V)\right] \equiv \log\left[\frac{\rho_0 |g|^{3/2}(x)}{\rho_t \sqrt{|g|}(E(x, v)) \det\left(\frac{\partial E}{\partial v}(x, v)\right)}\right].$$

From the identity (40), the map $\mathcal{V}$ given by

$$\mathcal{V}^i(x, y) = -g^{ij}(x)\ \partial_{x^j} c(x, y) \tag{74}$$

near $(0, y_0)$ (with $y_0 := y(p_0)$), satisfies:

$$y = E(x, v) \Longleftrightarrow v = \mathcal{V}(x, y),$$

and at $y_t := y[\exp_m(\mathrm{grad}_m u_t)]$, recalling (3), we get from (74):

$$\mathfrak{A}_{ij}(x, du_t) \equiv \partial^2_{x^i x^j} c(x, y)|_{y = y_t} .$$

So Equation (1) locally reads:

$$\log\det(w_{ij}) = \psi(x, \nabla_x u_t) \tag{75}$$

where $\nabla_x u_t := T_m x(\mathrm{grad}_m u_t) \equiv \mathcal{V}(x, y_t)$ and

$$w_{ij}\ dx^i \otimes dx^j := \mathrm{Hess}^{(c)} u_t \equiv \partial^2_{x^i x^j}[c(x,y) + u_t(x)]|_{y=y_t}\ .$$

B.2 Maximum principle à la Ma–Trudinger–Wang

Let us consider the test-function $m \mapsto \mathfrak{T}(m)$ on M_n equal to the g-trace of the covariant symmetric tensor $H_{ij}dx^i \otimes dx^j := \mathrm{Hess}^{(c)} u_t(m)$ and let $m_0 \in M_n$ be a point where $\mathfrak{T}$ assumes its maximum. We aim at a uniform upper bound on $\mathfrak{T}(m_0)$; since the tensor field $\mathrm{Hess}^{(c)} u_t$ is positive-definite, its eigenvalues with respect to the metric g will, indeed, be uniformly controlled by such a bound.

At the maximum point m_0, if $du_t(m_0) = 0$ we take a Riemannian normal chart [38]; if $du_t(m_0) \neq 0$, we take a Fermi chart along the vector $V_0 = \mathrm{grad}_{m_0} u_t$ as in Definition 1. In either case, we use the same chart x at m_0 (where x is centered) and at $p_0 = \exp_{m_0}(V_0)$, but it is convenient to stick to the (x, y) notation of [33], using y to denote the second argument of the local expression of the cost-function c, and to set still $y_0 = x(p_0)$ and $y_t := x[\exp_m(\mathrm{grad}_m u_t)]$, thus with $y_t = E(x, \nabla_x u_t)$ where $x = x(m)$. The test-function $\mathfrak{T}$ reads:

$$\mathfrak{T}(x) = g^{ij}(x)\partial^2_{x^i x^j}[c(x,y) + u_t(x)]|_{y=y_t}$$

near $x = 0$. Using Equation (75), one can now derive for $\mathfrak{T}$ at $x = 0$ an estimate which is a close variant of the quite robust one presented in [33, pp.162-164]. To do so, a careful inspection of the proof shows that, granted the existence of a positive lower bound θ on the c-curvatures at (m_0, V_0) as in (18), we require nothing but bounds under control on the second derivatives of the local tensor g^{ij} at $x = 0$ and on the local functions $\psi(x)$ and $c(x, y)$ together with the following derivatives of theirs:

$$\partial_{x^i}\psi, \partial_{v^j}\psi, \partial^2_{x^i x^j}\psi, \partial^2_{x^i v^j}\psi, \partial^2_{v^i v^j}\psi,$$

$$\partial^2_{x^i x^j}c, \partial^2_{x^i y^j}c, \partial^3_{x^i x^j x^k}c, \partial^3_{x^i x^j y^k}c, \partial^3_{x^i y^j y^k}c, \partial^4_{x^i x^j x^k x^l}c, \partial^4_{x^i x^j x^k y^l}c, \partial^4_{x^i x^j y^k y^l}c,$$

respectively calculated at $x = 0$ and at $(x, y) = (0, y_0)$. Granted such bounds, the proofs of Corollaries 1 and 2 are thus complete.

B.3 Bounds under control on g^{ij}, ψ and c

Control on derivatives of g^{ij}

The first partial derivatives of g^{ij} at 0 vanish in either types of chart (Riemann or Fermi); so we are left with the second derivatives, given by:

$$\partial_{kl} g^{ij}(0) = -\partial_{kl} g_{ij}(0).$$

In a Riemannian normal chart (if $V_0 = 0$), the derivatives $\partial_{li} g_{jk}(0)$ are intrinsic (formally given by the next equation), a result which goes back to Riemann's dissertation (see [38, chap.4]). In the Fermi chart case (if $V_0 \neq 0$), aside from (28), we require the classical identity, valid on the axis:

$$\partial_{\alpha\beta} g_{\gamma\lambda} = \frac{1}{3}\left(R^{\alpha}_{\gamma\lambda\beta} + R^{\alpha}_{\lambda\gamma\beta}\right) . \tag{76}$$

It can be checked (from the definition of the Riemann curvature tensor) by routine calculation, using (34) and (35).

Controls on E and ψ

If $V_0 = 0$, sticking to the notations of Section 2.2, we have $X(0,0,t) \equiv 0$ and:

$$\partial_{x^j} X^i(0,0,t) = \delta^i_j \ , \ \partial_{v^j} X^i(0,0,t) = t\delta^i_j \ ,$$

$$\forall (k,l) \in \mathbb{N}^2, k+l \geq 2 \Rightarrow |\partial^k_x \partial^l_v X(0,0,t)| \equiv 0.$$

Since $E(x,v) \equiv X(x,v,1)$, Lemma 14 below (read with $v_0 = 0$) follows at once. Furthermore, using the notation (67), we also get $Y^i_j \equiv \delta^i_j$ which, combined with the preceding result, readily yields the required bounds on ψ and its first and second derivatives at $x = 0$.
If $V_0 \neq 0$, since $u_t \in \mathcal{A}$, we have $(m_0, V_0) \in \text{NoCut}$. Of course, for (x,v) close to $(0, v_0)$, the identity $E(x,v) \equiv X(x,v,1)$ holds in the Fermi chart as well. Moreover, one can readily establish for $|\partial^2_{xx} X(0,v_0,t)|$ (resp. $|\partial^3_{xxv} X(0,v_0,t)|$) a boundedness result analogous to that of Lemma 9 (resp. Lemma 12). Combining the latter with Lemmas 4, 9, and 12, we infer the:

Lemma 14 *The g-norms of:*

$$\partial_x E, \partial_v E, \partial^2_{xx} E, \partial^2_{xv} E, \partial^2_{vv} E, \partial^3_{xxx} E, \partial^3_{xxv} E, \partial^3_{xvv} E, \partial^3_{vvv} E,$$

calculated at $(0, v_0)$, are under control.

Besides, recalling (67), we have: $[\partial_v E(0,v)]^{-1} \equiv Y$ for v close to v_0, and the bound (71) (together with $Y^n_a = Y^a_n = \delta_{na}$) combined with Lemma 14 yields again the required bounds on the function ψ and its first and second derivatives at $x = 0$. In the sequel, we thus focus on bounds for the sole function $c(x,y)$.

Control on $\mathcal{V}$

Recalling (74), setting for short $\mathcal{Y} = \mathcal{Y}(x,y) := \dfrac{\partial \mathcal{V}}{\partial y}$ and differentiating with respect to y the identity $y = E[x, \mathcal{V}(x,y)]$ (with x fixed), we find:

$$\mathcal{Y}^a_j \ \partial_{v^a} E^i = \delta^i_j \ ; \tag{77}$$

in particular, letting $x = 0$ and recalling (67), we may record at $y = E(0,v)$ the identity:

$$\mathcal{Y}(0,y) \equiv Y(v) \ .$$

Differentiating (77), once again with respect to y, yields:

$$\partial_{y^p} \mathcal{Y}^a_j = -\mathcal{Y}^b_p \mathcal{Y}^a_i \mathcal{Y}^c_j \ \partial^2_{v^b v^c} E^i \ . \tag{78}$$

Besides, differentiating with respect to x (for fixed v) the other identity, namely $v = \mathcal{V}[x, E(x,v)]$, we get:

$$\partial_{x^k} \mathcal{V}^a = -\mathcal{Y}^a_i \partial_{x^k} E^i \ . \tag{79}$$

Using the latter to differentiate (77) with respect to x, we obtain:

$$\partial_{x^k} \mathcal{Y}^a_j = -\mathcal{Y}^a_i \mathcal{Y}^b_j \left(\partial^2_{x^k v^b} E^i - \mathcal{Y}^c_l \ \partial^2_{v^b v^c} E^i \ \partial_{x^k} E^l\right) \ . \tag{80}$$

From $\partial_{y^k} \mathcal{V}^a \equiv \mathcal{Y}^a_k$ and (79) combined with (78) and (80), we readily infer the:

Lemma 15 *All the partial derivatives of $\mathcal{V}$ at (x,y) are expressible (in a polynomial way) solely in terms of $\dfrac{\partial \mathcal{V}}{\partial y}$ itself and the partial derivatives of E evaluated at $[x, \mathcal{V}(x,y)]$. In particular, the g-norm of the third order jet of $\mathcal{V}$ calculated at $(0, y_0)$ is under control.*

The final statement of the lemma simply follows from Lemma 14 combined with the bound (71). We are now in position to deal with the derivatives of the function $c(x,y)$.

Control on c

From (74) we get:

$$\partial_{x^j} c(x,y) = -g_{js}(x) \mathcal{V}^s(x,y) \ , \tag{81}$$

which yields successively, at $(0, y_0)$:

$$\partial^2_{x^j y^k} c = -\partial_{y^k} \mathcal{V}^j, \ \partial^3_{x^j y^k y^l} c = -\partial_{y^k y^l} \mathcal{V}^j,$$

$$\partial^2_{x^i x^j} c = -\partial_{x^i} \mathcal{V}^j, \ \partial^3_{x^i x^j y^k} c = -\partial_{x^i y^k} \mathcal{V}^j, \ \partial^4_{x^i x^j y^k y^l} c = -\partial_{x^i y^k y^l} \mathcal{V}^j,$$

hence, by Lemma 15, the preceding derivatives of c at $(0, y_0)$ are under intrinsic control. Next, since $\mathcal{V}^i(0, y_0) = \delta^i_n |v_0|$, further differentiating (81) provides us with a set of three equalities, beginning with:

$$\partial^3_{x^l x^i x^j} c(0, y_0) = -|v_0| \ \partial_{li} g_{nj}(0) - \partial^2_{x^l x^i} \mathcal{V}^j(0, y_0)$$

which shows, recalling (28) and Lemma 15, that the derivatives $\partial^3_{x^l x^i x^j} c(0, y_0)$ are under control. The second equality which we get is:

$$\partial^4_{x^l x^i x^j y^r} c(0, y_0) = -\partial_{li} g_{jk}(0) \ Y^k_r(v_0) - \partial^2_{x^l x^i} \mathcal{Y}^j_r(0, y_0)$$

Combining (28), (76) with Lemma 15 and the bound (71), we readily infer that the derivatives $\partial^4_{x^l x^i x^j y^r} c(0, y_0)$ are under control.
The final equality which we get is:

$$\partial^4_{x^k x^l x^i x^j} c(0, y_0) = -|v_0| \partial_{kli} g_{nj}(0) - \sum_{(k,l,i)} \partial_{kl} g_{rj}(0) \partial_{x^i} \mathcal{V}^r(0, y_0) - \partial^3_{x^k x^l x^i} \mathcal{V}^j(0, y_0)$$

the right-hand side of which is again under control for the same aforementioned reasons except for its $\partial_{kli} g_{nj}(0)$ term whenever all three indices k, l, i lie in $\{1, \ldots, n-1\}$. The terms $\partial_{\alpha\beta\gamma} g_{nn}(0)$ turn out to be controlled by (31) because they coincide with $2\partial_{\alpha\beta} \Gamma^n_{n\gamma}(0)$. As regards the others, noting the identity:

$$\partial_{\alpha\beta\gamma} g_{n\lambda} = \partial_{\alpha\beta} \left(\Gamma^\lambda_{n\gamma} + \Gamma^n_{\lambda\gamma}\right)$$

valid on the axis, and recalling (32), their control reduces to another one on $\partial_{\alpha\beta}\Gamma^n_{\lambda\gamma}(0)$, provided in Lemma 16 below.
Finally, in a Riemannian normal chart (case $v_0 = 0$), each of the previous controls holds *a fortiori*; the last one relies on the formula

$$\partial_{ikl} g_{js}(0) = \frac{1}{6}\nabla_i R_{jksl}(0)$$

which goes back to Elie Cartan [11, p.243, Eq.(21)] (see also [22, p.193]).

Lemma 16 *The following identity holds on the axis of a Fermi chart:*

$$\begin{aligned} \partial_{\alpha\beta}\Gamma^n_{\lambda\gamma} &= \frac{1}{4}\left(\nabla_\alpha R^\lambda_{\gamma n\beta} + \nabla_\beta R^\lambda_{\gamma n\alpha}\right) \\ &- \frac{1}{2}\nabla_\gamma\left(R^\lambda_{\alpha n\beta} + R^\lambda_{\beta n\alpha}\right) \\ &+ \frac{5}{12}\nabla_n\left(R^\lambda_{\alpha\gamma\beta} + R^\lambda_{\beta\gamma\alpha}\right) \ . \end{aligned} \tag{82}$$

Proof. For completeness, we first briefly recall the argument that leads to (33) read with $i = n$. In our Fermi chart, since $t \mapsto (tx^1, \ldots, tx^{n-1}, x^n)$ is a geodesic, we get using the geodesic equation:

$$\Gamma^i_{\alpha\beta}(x)x^\alpha x^\beta \equiv 0 \text{ and } \ g_{\alpha\beta}(x)x^\alpha x^\beta \equiv \sum_{\alpha=1}^{n-1}(x^\alpha)^2,$$

from what we readily infer:

$$x^\alpha x^\beta \partial_\alpha g_{n\beta}(x) \equiv 0.$$

The quantity $\varrho(x) := \sqrt{\sum_{\alpha=1}^{n-1}(x^\alpha)^2}$ represents the distance to the axis. Using cylindrical coordinates, the trick is now to apply to the latter equation the operator $\varrho\partial_\varrho \equiv x^\gamma\partial_\gamma$. It yields:

$$x^\alpha x^\beta x^\gamma \partial_{\alpha\gamma} g_{n\beta}(x) \equiv 0.$$

Setting $x^\alpha = \varrho\theta^\alpha$, dividing by ϱ^3 then letting $\varrho \downarrow 0$, we get at $x = (0, x^n)$ the identity (33) read with $i = n$ (since the unit vector $\theta^\alpha\partial_\alpha$ is arbitrary in the hyperplane orthogonal to the axis). The same argument repeated once yields on the axis the higher order identity (now with a circular summation on 4 indices):

$$\sum_{(\alpha,\beta,\gamma,\lambda)} \partial_{\alpha\beta\gamma} g_{n\lambda} \equiv 0 \ .$$

Combining it with (33), (34) and (35) enables one to check by brut calculation the following equality:

$$2\partial_{\alpha\beta\gamma} g_{n\lambda} = \nabla_n R^\lambda_{\alpha\gamma\beta} - \nabla_\gamma\left(R^\lambda_{\alpha n\beta} + R^\lambda_{\beta n\alpha}\right) - \nabla_\alpha R^\lambda_{\gamma n\beta}$$

valid on the axis, from which Lemma 16 routinely follows.

C Optimal transport regularity and covering spaces

The following result, yet unstated in the literature, is by now well-known:

Theorem 3 (folklore result) *Let $p : (\widetilde{M_n}, \widetilde{g}) \to (M_n, g)$ be a Riemannian normal (or Galoisian) covering map between compact connected n-dimensional manifolds; set Γ for its covering transformations group, thus a finite subgroup of isometries of $(\widetilde{M_n}, \widetilde{g})$. Let $(\widetilde{\mu_0}, \widetilde{\mu_1})$ be a couple of Γ-invariant smooth positive measures of same total mass on $\widetilde{M_n}$ and let (μ_0, μ_1) be the couple of associated smooth positive measures on M_n, which satisfy the Radon-Nikodym derivatives equality:*

$$\frac{d\widetilde{\mu_i}}{\widetilde{\mathrm{dVol}}} = \frac{d\mu_i}{\mathrm{dVol}} \circ p \tag{83}$$

where $i \in \{0, 1\}$. The optimal transportation map pushing $\widetilde{\mu_0}$ to $\widetilde{\mu_1}$ is smooth if and only if so is the optimal transportation map pushing μ_0 to μ_1.

Proof. Assume that the optimal transportation map $G = \exp(\mathrm{grad}\, u)$ pushing μ_0 to μ_1 is smooth. Setting $\widetilde{u} = p^* u$ and recalling that p is locally an isometry, naturality and geodesic uniqueness yield for the smooth map $\widetilde{G} := \exp(\mathrm{grad}\, \widetilde{u})$ the covering morphism relation:

$$p \circ \widetilde{G} = G \circ p \ ; \tag{84}$$

moreover, for each $\gamma \in \Gamma$, since the potential $\widetilde{u}$ is Γ-invariant and γ is an isometry, we have:

$$\gamma \circ \widetilde{G} = \widetilde{G} \circ \gamma \ . \tag{85}$$

For each measurable real function $\widetilde{f}$ on $\widetilde{M_n}$, set $\widetilde{f_\Gamma}$ for the Γ-invariant function obtained by averaging $\widetilde{f}$ over Γ:

$$\forall \widetilde{m} \in \widetilde{M_n},\ \widetilde{f_\Gamma}(\widetilde{m}) = \frac{1}{r} \sum_{\gamma \in \Gamma} \widetilde{f}[\gamma(\widetilde{m})]$$

where r stands for the cardinal of the deck group Γ (so the covering is r-sheeted); set f_Γ for the function on M_n defined by: $\widetilde{f_\Gamma} = p^* f_\Gamma$. The following identity clearly holds: $\int_{\widetilde{M}} \widetilde{f}\ \widetilde{\mathrm{dVol}} = \int_{\widetilde{M}} \widetilde{f_\Gamma} \widetilde{\mathrm{dVol}}$, hence also, from (83), the other one:

$$\int_{\widetilde{M}} \widetilde{f} d\widetilde{\mu_i} = \int_{\widetilde{M}} \widetilde{f_\Gamma} d\widetilde{\mu_i} \ . \tag{86}$$

Recalling $G_\# \mu_0 = \mu_1$, the latter with $i = 1$ yields:

$$\int_{\widetilde{M}} \widetilde{f} d\widetilde{\mu_1} = r \int_M f_\Gamma d\mu_1 = r \int_M (f_\Gamma \circ G) d\mu_0 = \int_{\widetilde{M}} (f_\Gamma \circ G \circ p) d\widetilde{\mu_0} \ .$$

Using (84)(85), we get: $\int_{\widetilde{M}} \widetilde{f} d\widetilde{\mu_1} = \int_{\widetilde{M}} (\widetilde{f_\Gamma} \circ \widetilde{G}) d\widetilde{\mu_0} = \int_{\widetilde{M}} (\widetilde{f} \circ \widetilde{G})_\Gamma \ d\widetilde{\mu_0}$ and by (86) we obtain:

$$\int_{\widetilde{M}} \widetilde{f} d\widetilde{\mu_1} = \int_{\widetilde{M}} (\widetilde{f} \circ \widetilde{G}) d\widetilde{\mu_0} \ .$$

Since $\widetilde{f}$ is arbitrary, it means that the map $\widetilde{G}$ pushes the measure $\widetilde{\mu_0}$ to $\widetilde{\mu_1}$; besides, the map $\widetilde{G}$ is optimal, unique [34] and smooth, so the first part of the

equivalence is proved.
Conversely, let the smooth map $\widetilde{G} = \exp(\operatorname{grad}\widetilde{u})$ push $\widetilde{\mu_0}$ to $\widetilde{\mu_1}$. So must do the map $\exp[\operatorname{grad}(\widetilde{u}\circ\gamma)]$, for each $\gamma \in \Gamma$, since γ is an isometry which preserves the $\widetilde{\mu_i}$'s. By uniqueness of the potential $\widetilde{u}$ (up to an additive constant) [16], the function $\widetilde{u}$ must be Γ-invariant as well. Let u be the function on M_n defined by $\widetilde{u} = p^*u$ (and μ_i the measure on M_n defined by (83)). Consider the smooth map $G = \exp(\operatorname{grad} u)$; the relation (84) is again satisfied. Moreover, using $\widetilde{G}_\#\widetilde{\mu_0} = \widetilde{\mu_1}$, we find for each measurable function f on M_n:

$$\int_M f d\mu_1 = \frac{1}{r}\int_{\widetilde{M}}(f\circ p)d\widetilde{\mu_1} = \frac{1}{r}\int_{\widetilde{M}}(f\circ p\circ \widetilde{G})d\widetilde{\mu_0} \ .$$

From (84), we further get:

$$\int_M f d\mu_1 = \frac{1}{r}\int_{\widetilde{M}}(f\circ G\circ p)d\widetilde{\mu_0} = \int_M (f\circ G)d\mu_0 \ ,$$

or else, since f is arbitrary: $G_\#\mu_0 = \mu_1$. The proof of Theorem 3 is complete.

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